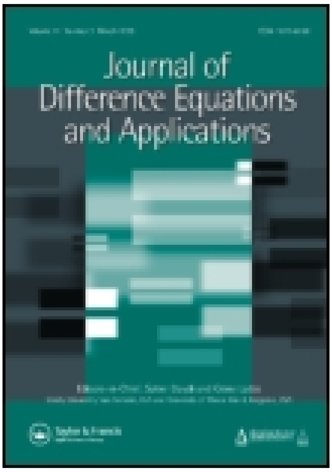

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Mappings with a single critical point and applications to rational difference equations

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Mappings with a single critical point and applications to rational difference equations

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Convergence properties of first-order difference equations of the form $x_{n+1} = f(x_n)$ are established for a general class of mappings f, where f has at most one critical point. Using these results, we find necessary and sufficient conditions for the convergence of the solutions for all difference equations of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_n + C}{\alpha x_n^2 + \beta x_n + \gamma}$$

for all possible choices of non-negative coefficients and positive initial values.

Keywords: first-order map; rational difference equations; convergence; single critical point

1. Introduction

This paper concerns first-order difference equations $x_{n+1} = f(x_n)$ for mappings f on the line. Such equations can also be regarded as one-dimensional iterating maps. As an iterating map, many results have been established. For example, when f is quadratic or logistic, the dynamic behaviour of the map has been well studied. One can find related results in the book [15]. There are also studies for more general maps, for example [2,8]. In the paper [16], by comparing a continuous function f with its inverse f^{-1} in a neighbourhood of an isolated fixed point $\bar{x}$, a necessary and sufficient condition for the asymptotic stability of $\bar{x}$ is obtained.

There has been a great deal of interest in difference equations when the mapping f is rational. In particular, when f is linear in the numerator and denominator, the dynamics of the solutions have been extensively studied. Necessary and sufficient conditions concerning the behaviour of their solutions have been obtained, see [5,9–11]. However, as soon as nonlinear factors are introduced into either the numerator or the denominator, the results are less complete. In papers [13,14], high-order rational difference equations are studied, where the numerator is quadratic and the denominator is linear. The first-order difference equation can be regarded as a special case of their study. Sufficient conditions were established on the parameter values which guarantee that the unique non-negative fixed point attracts all positive solutions. One can also find other results for high-order rational difference equations whose numerator is quadratic and denominator is linear or quadratic. See papers [1,6,7], and book [3] among others. In paper [4], the authors study the convergence of second-order rational difference equations with quadratic terms. With a transformation, their equations can be reduced to first-order linear-quadratic rational

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difference equations and the results in that paper can be applied. When f is a quadratic-quadratic rational function, there are partial results on the convergence properties of this equation (see [12]).

In this paper, we consider first-order difference equations $x_{n+1} = f(x_n)$ when function f is C^1 with at most one critical point. We completely determine the convergence properties of the solution to this difference equation for any positive initial values if either the difference equation has no prime period-two solutions or all of the prime period-two solutions lie on one particular side of the critical point, depending upon whether the critical point gives a maximum or a minimum.

As an application of our results, we consider the first-order quadratic-quadratic rational difference equation

$$x_{n+1} = \frac{Ax_n^2 + Bx_n + C}{\alpha x_n^2 + \beta x_n + \gamma}, \tag{1}$$

where the coefficients A, B, C, α, β and γ are non-negative. We find necessary and sufficient conditions for the convergence of solutions of this equation for all possible non-negative coefficients and positive initial values.

2. Definitions and background material results

In order to prove the main theorems, we first establish a number of preliminary results, several of which are interesting in their own right.

Definition 2.1. *A real number $\bar{x}$ is said to be a fixed point of the function $f : R \rightarrow R$ if and only if $f(\bar{x}) = \bar{x}$.*

Definition 2.2. *A function $f : R \rightarrow R$ is said to have a (positive) prime period-two pair $(\bar{p}_1, \bar{p}_2)$, if and only if there exists (positive) real numbers $\bar{p}_1$ and $\bar{p}_2$, with $\bar{p}_1 \neq \bar{p}_2$, such that $\bar{p}_2 = f(\bar{p}_1)$ and $\bar{p}_1 = f(\bar{p}_2)$. We call $\bar{p}_1$ and $\bar{p}_2$ prime period-two points of* f.

Definition 2.3. *We define the* n*-th iterate of* f *as $f^{n+1}(x) = f(f^n(x))$ for any integer $n \geq 1$ and $f^1(x) = f(x)$.*

Observe that $\bar{x}$ is a positive fixed point of f^2 if and only if $\bar{x}$ is a positive fixed point of f *or a positive prime period-two point of* f.

Definition 2.4. *We say that $\{f^n(x_0)\} \equiv \{x_0, f(x_0), f^2(x_0), \ldots\}$ is the solution of $x_{n+1} = f(x_n)$ with initial value x_0.*

Definition 2.5. *The solution $\{f^n(x_0)\}$ is said to converge to the real number $\bar{x}$ if and only if $\lim_{n\to\infty} f^n(x_0) = \bar{x}$.*

Definition 2.6. *The solution $\{f^n(x_0)\}$ is said to converge to the prime period-two pair $(\bar{p}_1, \bar{p}_2)$ if and only if either $\lim_{n\to\infty} f^{2n}(x_0) = \bar{p}_1$ and $\lim_{n\to\infty} f^{2n+1}(x_0) = \bar{p}_2$, or $\lim_{n\to\infty} f^{2n}(x_0) = \bar{p}_2$ and $\lim_{n\to\infty} f^{2n+1}(x_0) = \bar{p}_1$.*

LEMMA 2.7. *Suppose that* $f \in C((0,\infty)\times(0,\infty))$. *Any solution to* $x_{n+1}=f(x_n)$ *that converges to a prime period-two pair must be a positive prime period-two pair.*

Proof. We need to rule out the possibility that the prime period-two pair is of the form $(\bar{p},0)$ or $(0,\bar{p})$ with $\bar{p}>0$. Suppose that $\lim_{n\to\infty} f^{2n}(x_0)=\bar{p}$ and $\lim_{n\to\infty} f^{2n+1}(x_0)=0$. Since $\bar{p}>0$ and f is continuous on $(0,\infty)$, then $0=\lim_{n\to\infty} f^{2n+1}(x_0)=\lim_{n\to\infty} f(f^{2n}(x_0))=f(\bar{p})$. This is impossible since $f(x)>0$ for $x>0$. By a similar argument, the solutions cannot converge to the prime period-two pair $(0,\bar{p})$.

We will say that 0 is a fixed point of a function f if f can be extended continuously to the origin so that $f(0)=0$. Otherwise, we will say that 0 is not a fixed point of f. □

LEMMA 2.8. *Suppose* $f \in C((0,\infty)\times(0,\infty))$.

(a) *If* $f((0,\alpha]) \subseteq (0,\alpha]$ *for some* $\alpha>0$ *and* 0 *is not a fixed point of* f^2, *then for every* $x_0 \in (0,\alpha]$, *the solution* $\{f^n(x_0)\}$ *converges to a positive fixed point of* f *or a positive prime period-two pair of* f *in the interval* $(0,\alpha]$ *if and only if the solution* $\{f^{2n}(x_0)\}$ *converges to a fixed point of* f^2 *in the interval* $(0,\alpha]$.
(b) *If* 0 *is not a fixed point of* f^2, *then for every* $x_0 \in (0,\infty)$, *the solution* $\{f^n(x_0)\}$ *converges to a positive fixed point of* f *or a positive prime period-two pair of* f *in the interval* $(0,\infty)$ *if and only if the solution* $\{f^{2n}(x_0)\}$ *converges to a positive fixed point of* f^2 *in the interval* $(0,\infty)$.
(c) *If* $f([\alpha,\beta]) \subseteq [\alpha,\beta]$ *with* $0<\alpha<\beta$, *then for every* $x_0 \in [\alpha,\beta]$, *the solution* $\{f^n(x_0)\}$ *converges to a fixed point of* f *or a prime period-two pair of* f *in the interval* $[\alpha,\beta]$ *if and only if the solution* $f^{2n}(x_0)$ *converges to a fixed point of* f^2 *in the interval* $[\alpha,\beta]$.
(d) *If* $f([\alpha,\infty)) \subseteq [\alpha,\infty)$ *with* $\alpha>0$, *then for every* $x_0 \in [\alpha,\infty)$, *the solution* $\{f^n(x_0)\}$ *converges to a fixed point of* f *or a prime period-two pair of* f *in the interval* $[\alpha,\infty)$ *if and only if the solution* $\{f^{2n}(x_0)\}$ *converges to a fixed point of* f^2 *in the interval* $[\alpha,\infty)$.
(e) *If* f *can be extended to be continuous at the origin, then for every* $x_0 \in [0,\infty)$, *the solution* $\{f^n(x_0)\}$ *converges to a non-negative fixed point of* f *or a positive prime period-two pair of* f *in the interval* $[0,\infty)$ *if and only if the solution* $\{f^{2n}(x_0)\}$ *converges to a fixed point of* f^2 *in the interval* $[0,\infty)$.

The proof of this lemma is trivial, and we dispense with the proof.

THEOREM 2.9. *Suppose* $x_{n+1}=f(x_n)$, *where*

(i) $f \in C([\alpha,\infty)\times[\beta,\infty))$ *with* $f(\alpha)=\beta\geq\alpha\geq 0$,
(ii) f *is strictly increasing on* $[\alpha,\infty)$, *and*
(iii) f *has at most a finite number of fixed points in the interval* $[\alpha,\infty)$.

Let $\bar{x}_{\sup}=\max\{\bar{x} : \bar{x}$ *is a non-negative fixed point of* $f\}$. *Then for every initial value* $x_0\geq\alpha$, *the solution* $\{f^n(x_0)\}$ *converges to one of the fixed points of* f *in the interval* $[\alpha,\infty)$ *in the case* $f(x)<x$ *as* $x\to\infty$. *For the case* $f(x)>x$ *as* $x\to\infty$, *we conclude that the solution* $\{f^n(x_0)\}$ *converges to a fixed point of* f *when* $\alpha\leq x_0\leq\bar{x}_{\sup}$ *and diverge to infinity when* $x_0>\bar{x}_{\sup}$.

This result is well known and can be proved using the elementary graphic method of iteration. This technique can be found in introductory texts on dynamical systems (for example, see Section 9.2 of [15]).

Assume that $0 \le \alpha_1 < \alpha_2 < \infty$ and $0 \le \beta_1 < \beta_2 < \infty$.

THEOREM 2.10. *Suppose $x_{n+1} = f(x_n)$, where*

(i) $f \in C([\alpha_1, \alpha_2]) \times [\beta_1, \beta_2])$ *with* $f(\alpha_1) = \beta_1 \ge \alpha_1, f(\alpha_2) = \beta_2 \le \alpha_2$,
(ii) *f is strictly increasing on $[\alpha_1, \alpha_2]$, and*
(iii) *f has at most a finite number of fixed points in the interval $[\alpha_1, \alpha_2]$.*

Then for all initial values $x_0 \in [\alpha_1, \alpha_2]$, the solution $\{f^n(x_0)\}$ converges to one of the fixed points of f in the interval $[\alpha_1, \alpha_2]$.

Once again, this result is well known and follows using the elementary graphic method of iteration.

3. The main general results

THEOREM 3.1. *Suppose the function $f \in C((0, \infty) \times (0, \infty))$, f is strictly decreasing, and f has at most a finite number of positive prime period-two pairs. Then we have the following conclusion. For every $x_0 > 0$ the solution $\{f^n(x_0)\}$ converges to either the unique positive fixed point of f or to a positive prime period-two pair of f, except for the case that all of the following three conditions hold: $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$ and $f^2(x) < x$ for $x > 0$ and x sufficiently small. Denote $\bar{x}_{\min} = \min\{\bar{x} > 0 : \bar{x}$ is either a fixed point of f or a prime period-two point of $f\}$. In the case that all of the three additional conditions above hold, we have the following conclusion:*

(i) *For $0 < x_0 < \bar{x}_{\min}$, we have $f^{2n}(x_0) \to 0$ and $f^{2n+1}(x_0) \to \infty$ as $n \to \infty$,*
(ii) *for $\bar{x}_{\min} \le x_0 \le f(\bar{x}_{\min})$, the solution $\{f^n(x_0)\}$ converges either to the unique positive fixed point of f or to a positive prime period-two pair of f in the interval $[\bar{x}_{\min}, f(\bar{x}_{\min})]$, and*
(iii) *for $x_0 > f(\bar{x}_{\min})$, we have $f^{2n}(x_0) \to \infty$ and $f^{2n+1}(x_0) \to 0$ as $n \to \infty$,*

Proof. We first claim that f^2 is strictly increasing. Consider any $0 < x_1 < x_2$. Since f is strictly decreasing, then $f(x_2) < f(x_1)$ and so $f^2(x_2) > f^2(x_1)$ as claimed. Since f is continuous on $(0, \infty)$ and the range of f is $(0, \infty)$, then clearly f^2 is continuous on $(0, \infty)$. Since f^2 is strictly increasing, then we can extend f^2 so that it is continuous on $[0, \infty)$ with its range in $[0, \infty)$. Since f is strictly decreasing, it has exactly one fixed point $\bar{x}$ and this fixed point is positive. Since f is strictly decreasing, we conclude that either $0 < \lim_{x\to 0} f(x) < \infty$ or $\lim_{x\to 0} f(x) = \infty$.

Case 3.1.1. Assume that $0 < \lim_{x\to 0} f(x) < \infty$.

In this case f can be extended to be continuous on $[0, \infty)$. Since f is strictly decreasing, then $\lim_{x\to\infty} f(x) = a$ for some finite $a \ge 0$. This gives $\lim_{x\to\infty} f^2(x) = \lim_{x\to a} f(x) = f(a) < \infty$. Thus, $f^2(x) < x$ as $x \to \infty$. By Theorem 2.9, for every $x_0 > 0$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. By assumption, $\lim_{x\to 0} f(x) < \infty$ and so $\lim_{x\to 0} f(x) = b$ for some finite $b > 0$. Thus, $\lim_{x\to 0} f^2(x) = \lim_{x\to b} f(x) = f(b) > 0$. Thus 0 is not a fixed point of f^2. By Lemma 2.8(b), every solution of f converges to either a positive fixed point of f or a positive prime period-two pair of f.

Case 3.1.2. Assume that $\lim_{x\to 0} f(x) = \infty$.

Case 3.1.2.1. Assume that $\lim_{x\to\infty} f(x) = b$ with $b > 0$.

Clearly b must be finite since f is strictly decreasing. Observe that $\lim_{x\to 0} f^2(x) = \lim_{x\to\infty} f(x) = b$. Thus, 0 is not a fixed point of f^2. Next, we have $\lim_{x\to\infty} f^2(x) = \lim_{x\to b} f(x) = f(b)$ which is obviously finite since f is strictly decreasing. Thus $f^2(x) < x$ as $x \to \infty$. By the same reasoning as in Case 3.1.1, every solution of f converges to either a positive fixed point of f or a positive prime period-two pair of f.

Case 3.1.2.2. Assume that $\lim_{x\to\infty} f(x) = 0$.

First, note that $\lim_{x\to 0} f^2(x) = \lim_{x\to\infty} f(x) = 0$. Thus, 0 is a fixed point of f^2. Consider $\bar{x}$, where $\bar{x}$ is either a fixed point of f or prime period-two point of f. Without loss of generality, we have $\bar{x} \leq f(\bar{x})$, since $\bar{x} = f(\bar{x})$ in the case $\bar{x}$ is a fixed point of f and otherwise $(\bar{x}, f(\bar{x}))$ is a prime period-two pair of f. We claim that $\bar{x}_{\min} \leq \bar{x} \leq f(\bar{x}) \leq f(\bar{x}_{\min})$. Since $\bar{x}_{\min} \leq \bar{x}$ and f is strictly decreasing, then $f(\bar{x}_{\min}) \geq f(\bar{x})$ as claimed. Thus, every fixed point of f or prime period-two point of f lies in the interval $[\bar{x}_{\min}, f(\bar{x}_{\min})]$. By Theorem 2.10, if $x_0 \in [\bar{x}_{\min}, f(\bar{x}_{\min})]$, then $f^{2n}(x_0)$ converges to a fixed point of f^2 in this interval. Since f is strictly decreasing in the interval $[\bar{x}_{\min}, f(\bar{x}_{\min})]$ and $f^2(\bar{x}_{\min}) = \bar{x}_{\min}$, then $f([\bar{x}_{\min}, f(\bar{x}_{\min})]) = [\bar{x}_{\min}, f(\bar{x}_{\min})]$. By Lemma 2.8(c), for every $x_0 \in [\bar{x}_{\min}, f(\bar{x}_{\min})]$, the solutions $f^n(x_0)$ converges either to a positive fixed point of f or to a positive prime period-two pair in this interval.

The convergence properties of the solutions on the intervals $(0, \bar{x}_{\min})$ and $(f(\bar{x}_{\min}), \infty)$, depend upon the behaviour of f^2 near the origin. So, we need two more subcases.

Case 3.1.2.2.1. Assume that $f^2(x) > x$ for $x > 0$ and x is sufficiently small.

Since 0 and $\bar{x}_{\min}$ are the only fixed points of f^2 on the interval $[0, \bar{x}_{\min}]$, then $f^2(x) > x$ for $0 < x < \bar{x}_{\min}$. By the elementary graphic method of iteration, for every $x_0 \in (0, \bar{x}_{\min})$, the solution $\{f^{2n}(x_0)\}$ converges to $\bar{x}_{\min}$. Since f is continuous on the interval $(0, \bar{x}_{\min}]$ and $f((0, \bar{x}_{\min}]) = [f(\bar{x}_{\min}), \infty)$, then for every $y \in [f(\bar{x}_{\min}), \infty)$, there exists $x \in (0, \bar{x}_{\min}]$ such that $f(x) = y$. Since f^2 is strictly increasing and $\bar{x}_{\min}$ is a fixed point of f^2, then $f^2((0, \bar{x}_{\min})) \subseteq (0, \bar{x}_{\min})$. Since $f^2(x) > x$ for $x \in (0, \bar{x}_{\min})$, $f^2((0, \bar{x}_{\min})) \subseteq (0, \bar{x}_{\min})$, and f is strictly decreasing, then $f^2(y) = f^2(f(x)) = f(f^2(x)) < f(x) = y$ for all $y \in [f(\bar{x}_{\min}), \infty)$. It follows by using the elementary graphic method of iteration that for every $x_0 \in (f(\bar{x}_{\min}), \infty)$, the solution $\{f^{2n}(x_0)\}$ converges to $f(\bar{x}_{\min})$. If $x_0 \in (0, \bar{x}_{\min})$, then since $x_0 < \bar{x}_{\min}$ and f is strictly decreasing, we have $f(x_0) > f(\bar{x}_{\min})$. Thus $\{f^{2n+1}(x_0)\}$ converges to $f(\bar{x}_{\min})$. If $f(\bar{x}_{\min}) = \bar{x}_{\min}$, then $\{f^n(x_0)\}$ converges to the unique fixed point of f. If $f(\bar{x}_{\min}) \neq \bar{x}_{\min}$, then $\{f^n(x_0)\}$ converges to the prime period-two pair $(\bar{x}_{\min}, f(\bar{x}_{\min}))$. If $x_0 \in (\bar{x}_{\min}, \infty)$, then by the same argument, $\{f^n(x_0)\}$ converges to $\bar{x}_{\min}$ if $\bar{x}_{\min}$ is the unique fixed point of f. If $f(\bar{x}_{\min}) \neq \bar{x}_{\min}$, then $\{f^n(x_0)\}$ converges to the prime period-two pair $(f(\bar{x}_{\min}), \bar{x}_{\min})$.

Case 3.1.2.2.2. Assume that $f^2(x) < x$ for $x > 0$ and x sufficiently small.

Since 0 and $\bar{x}_{\min}$ are the only fixed points of f^2 on the interval $[0, \bar{x}_{\min}]$, then $f^2(x) < x$ for $0 < x < \bar{x}_{\min}$. By the elementary graphic method of iteration, for every $x_0 \in (0, \bar{x}_{\min})$, the solution $\{f^{2n}(x_0)\}$ converges to 0. By the same argument as in Case 3.1.2.2.1, we have $f^2(y) > y$ for all $y \in (f(\bar{x}_{\min}), \infty)$. It follows by the elementary graphic method of iteration that for every $x_0 \in (f(\bar{x}_{\min}), \infty)$, the solution $\{f^{2n}(x_0)\}$ diverges to infinity. As in Case 3.1.2.2.1, if $x_0 \in (0, \bar{x}_{\min})$, then $f(x_0) \in (\bar{x}_{\min}, \infty)$. Thus, if $x_0 \in (0, \bar{x}_{\min})$, then $\{f^{2n}(x_0)\}$ converges to 0 and $\{f^{2n+1}(x_0)\}$ diverges to infinity. Similarly, if $x_0 \in (f(\bar{x}_{\min}), \infty)$, then $\{f^{2n}(x_0)\}$ diverges to infinity and $\{f^{2n+1}(x_0)\}$ converges to 0. □

Definition 3.2. *A C^1 function f is said to have a critical point at $x = \mu$ if and only if $f'(\mu) = 0$.*

In Theorem 3.3, denote $\bar{x}_{\max} = \max\{\bar{x} > 0 : \bar{x}$ is either a fixed point of f or a prime period-two point of $f\}$. Denote m to be the solution to $f(m) = \bar{x}_{\max}$ with $0 < m < \mu$, if it exists. We will see in the proof of Theorem 3.3 that if m exists, it must be unique. If m does not exist, choose $m = 0$.

THEOREM 3.3. *Suppose the function f has the following properties:*

(i) $f \in C^1((0,\infty) \times (0,\infty))$.
(ii) *f has at most a finite number of positive fixed points and positive prime period-two pairs.*
(iii) *f has exactly one relative extreme point at some $x = \mu$ with $\mu > 0$.*
(iv) *In the case of a relative maximum for f, every prime period-two point $\bar{p}$ of f satisfies $\bar{p} > \mu$. In the case of a relative minimum for f, every prime period-two point $\bar{p}$ of f satisfies $\bar{p} < \mu$.*

We have the following conclusions:

(a) In the event that $f(x) < x$ as $x \to \infty$, then for every initial value $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges either to a non-negative fixed point of f or to a positive prime period-two pair of f.
(b) In the event that $f(x) > x$ as $x \to \infty$, then for $x_0 \in [m, \bar{x}_{\max}]$, the solution $\{f^n(x_0)\}$ converges either to a fixed point of f or to a positive prime period-two pair of f in the interval $[m, \bar{x}_{\max}]$. When $x_0 \in (0,m) \bigcup (\bar{x}_{\max}, \infty)$, the solution $\{f^n(x_0)\}$ diverges to infinity. If f has no fixed points, then every solution diverges to infinity.

Proof.
Before conducting the details of this proof, we outline the main ideas. First, we point out that it is sufficient to study the convergence properties of f^2 since the fixed points of f^2 (with the possible exception of 0) are either the fixed points of f or the positive prime period-two points of f. An important element of the proof is that if the function f has a single critical point, then the function f^2 has all of its fixed points contained within either one or two intervals in which f^2 is strictly increasing. The elementary graphic method of iteration can easily prove convergence properties of solutions when the function is strictly increasing. On the intervals where f^2 is not strictly increasing we will be able to show that within a finite number k of iterations, $f^{2k}(x_0)$ will be in an interval where f^2 is strictly increasing. We will also show that for every initial value $x_0 \in (0,\infty)$, either x_0 or $f(x_0)$ will lie in an invariant interval I such that $f(I) \subseteq I$, where we will have been able to prove the convergence properties of $f^{2n}(x_0)$ for any $x_0 \in I$ as outlined above. This last step insures that we will also be able to handle the convergence properties of $f^{2n+1}(x_0)$. Now we give the details of the proof.

Since f is a C^1 function, then the relative extreme point of f at μ must be the unique critical point, that is: $f'(\mu) = 0$. The critical points of f^2 must satisfy $(f^2)'(x) = f'(f(x))f'(x) = 0$, so that they occur exactly when $x = \mu$ and at any x-values such that $f(x) = \mu$. Since f has exactly one relative extreme point, then there are at most two positive solutions to $f(x) = \mu$, say μ_1 and μ_2, with $\mu_1 < \mu_2$. Furthermore the fact that μ is the unique relative extreme point of f implies $\mu_1 < \mu < \mu_2$.

Case 3.3.1. The relative extreme point of f is a relative minimum.

Since f has exactly one critical point at $x = \mu$ and it is a relative minimum, then f is strictly decreasing for $x \leq \mu$ and strictly increasing for $x \geq \mu$. This implies that f has an absolute minimum at $x = \mu$.

Case 3.3.1.1. Assume that $f(\mu) \geq \mu$.

Case 3.3.1.1.1. Assume that $f(x) < x$ as $x \longrightarrow \infty$.

Since f is strictly decreasing for $x \leq \mu$, then $f(x) > f(\mu) \geq \mu$ for all $x < \mu$. Thus, if $x_0 \in (0, \mu)$, we have $f(x_0) > \mu$. Without loss of generality, $x_0 \in [\mu, \infty)$. Since f is strictly increasing on the interval $[\mu, \infty)$ and $f(\mu) \geq \mu$, we may apply Theorem 2.9 to obtain the conclusion that for every $x_0 \in [\mu, \infty)$, the solution $\{f^n(x_0)\}$ converges to either a fixed point of f or a positive prime period-two pair in this interval.

Case 3.3.1.1.2. Assume that $f(x) > x$ as $x \longrightarrow \infty$.

Note that if f has no fixed points, then $f(x) >$ for all $x > 0$. Clearly, for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ diverges to infinity. So assume that f has at least one fixed point, and thus $\bar{x}_{\max}$ must exist.

Case 3.3.1.1.2.1. Assume that there is a value m, $0 < m < \mu$, such that $f(m) = \bar{x}_{\max}$.

Since f is strictly decreasing on $(0, \mu]$ and strictly increasing on $[\mu, \infty)$, then the only intervals in which $f(x) > \bar{x}_{\max}$ are $(0, m)$ and $(\bar{x}_{\max}, \infty)$. Since $f(x) > x$ for $x \in (\bar{x}_{\max}, \infty)$, then the solution $\{f^n(x_0)\}$ diverges to infinity for $x_0 \in (0, m) \bigcup (\bar{x}_{\max}, \infty)$. In the interval $[m, \bar{x}_{\max}]$, observe that the absolute maxima of f occur at m and $\bar{x}_{\max}$, and the absolute minimum of f occurs at μ. Since $f(m) = f(\bar{x}_{\max})$ and $f(\mu) \geq \mu > m$, then $f([m, \bar{x}_{\max}]) \subseteq [\mu, \bar{x}_{\max}] \subset [m, \bar{x}_{\max}]$. Since f is strictly increasing on $[\mu, \bar{x}_{\max}], f(\mu) \geq \mu$, and $f(\bar{x}_{\max}) = \bar{x}_{\max}$, then by Theorem 2.10, for every $x_0 \in [m, \bar{x}_{\max}]$, the solution $\{f^n(x_0)\}$ converges either to a fixed point of f or to a prime period-two pair in this interval.

Case 3.3.1.1.2.2. Assume that there is no value m, $0 < m < \mu$, such that $f(m) = \bar{x}_{\max}$.

Since f is strictly increasing on $(\bar{x}_{\max}, \infty)$ and $f(x) > x$ for $x \in (\bar{x}_{\max}, \infty)$, then the solution $\{f^n(x_0)\}$ diverges to infinity for every $x_0 \in (\bar{x}_{\max}, \infty)$. Since f is strictly decreasing on $(0, \mu]$ and there is no solution to $f(m) = \bar{x}_{\max}$ with $0 < m < \mu$, then $0 < f(x) < \bar{x}_{\max}$ for $x \in (0, \mu]$. Since f is strictly decreasing on $(0, \mu]$ and $f(\mu) \geq \mu$, then $f(x_0) \in [\mu, \bar{x}_{\max}]$ for $x_0 \in (0, \mu)$. Thus, we only need to handle $x_0 \in [\mu, \bar{x}_{\max}]$. Since f is strictly increasing on $[\mu, \bar{x}_{\max}], f(\mu) \geq \mu$, and $f(\bar{x}_{\max}) = \bar{x}_{\max}$, then for every $x_0 \in [\mu, \bar{x}_{\max}]$, the solution $\{f^n(x_0)\}$ converges to a fixed point of f in this interval by Theorem 2.10.

Case 3.3.1.2. Assume that $f(\mu) < \mu$.

Figure 1 illustrates the graph of f for this case and, in particular, the Case 3.3.1.2.1.

Since f is strictly decreasing on the interval $(0, \mu)$, there is exactly one fixed point of f, say $\bar{x}$, in the interval $(0, \mu)$. Suppose that μ_1 exists (defined just before Case 3.3.1). We claim that $\mu_1 < \bar{x}$. We have $f(\mu_1) = \mu > \bar{x} = f(\bar{x})$. Since f is strictly decreasing on the interval $(0, \mu)$, then $\mu_1 < \bar{x}$ as claimed.

We also claim that any prime period-two point of f must lie in the interval (μ_1, μ). By assumption any prime period-two pair of f, $(\bar{p}_1, \bar{p}_2)$ and we choose $\bar{p}_1 < \bar{p}_2$, must satisfy $0 < \bar{p}_1 < \bar{p}_2 < \mu$. Since f is strictly decreasing on $(0, \mu], 0 < \mu_1 < \mu$, and $f(\mu_1) = \mu$, then $f(x) \geq \mu$ for $x \leq \mu_1$. This shows that $f(\bar{p}_1) = \bar{p}_2 < \mu$ implies $\bar{p}_1 > \mu_1$. Thus, $\mu_1 < \bar{p}_1 < \bar{p}_2 < \mu$, proving the claim.

Suppose μ_2 exists (defined just before Case 3.3.1). If f has any other fixed points besides $\bar{x}$, we claim that they must lie in the interval (μ_2, ∞). Let $\bar{x}' > \mu$ denote any another fixed point of f, if it exists. We have $f(\bar{x}') = \bar{x}' > \mu = f(\mu_2)$. Since f is strictly increasing on the interval (μ, ∞), then we must have $\bar{x}' > \mu_2$ as claimed.

Case 3.3.1.2.1. There are exactly two positive values say $x = \mu_1$ and $x = \mu_2$ such that $f(x) = \mu$, where we take $\mu_1 < \mu_2$.

We now consider the graph of f^2. Figure 2 illustrates the graph of f^2 for this case.

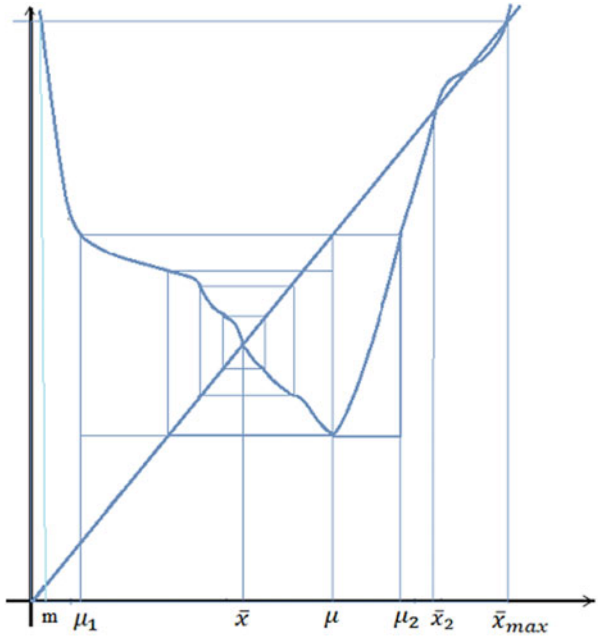

Figure 1. Graph of $y = f(x)$.

Recall that f^2 has exactly three critical points at μ, μ_1 and μ_2. Since $f(\mu)$ is the absolute minimum value of f, then by the assumptions of Case 3.3.1.2.1, the absolute minima of f^2 must occur exactly at μ_1 and μ_2 with $f^2(\mu_1) = f(\mu)$ and $f^2(\mu_2) = f(\mu)$. The remaining critical point μ of f^2 must be a relative maximum for f^2. Thus f^2 is strictly decreasing on the intervals $(0, \mu_1]$ and $[\mu, \mu_2]$, and f^2 is strictly increasing on the intervals $[\mu_1, \mu]$ and $[\mu_2, \infty)$.

We saw earlier that all of the fixed points of f^2 are greater than μ_1. Since f^2 is strictly decreasing on $(0, \mu_1]$ and f^2 has no fixed points in $(0, \mu_1]$, then $f^2(x) > x$ for $x \in (0, \mu_1]$.

Also note that $f^2(\mu_2) = f(\mu) < \mu < \mu_2$. Let $\bar{x}_2$ denote the smallest fixed point of f besides $\bar{x}$. If $\bar{x}_2$ does not exist, then define $\bar{x}_2 = \infty$. We saw earlier that $\bar{x}_2 > \mu_2$. Since $f^2(\mu_2) < \mu_2$ and f^2 has no fixed points in the interval $[\mu, \bar{x}_2)$, then $f^2(x) < x$ for $x \in [\mu, \bar{x}_2)$.

Let us summarize the properties of f^2 :

(i) f^2 is strictly decreasing on the intervals $(0, \mu_1]$ and $[\mu, \mu_2]$.
(ii) f^2 is strictly increasing on the intervals $[\mu_1, \mu]$ and $[\mu_2, \infty)$.
(iii) f^2 has absolute minima at $x = \mu_1$ and $x = \mu_2$, and $f^2(\mu_1) = f^2(\mu_2) = f(\mu)$.
(iv) f^2 has a relative maximum at $x = \mu$.
(v) The smallest positive fixed point of f occurs at $x = \bar{x}$ with $\mu_1 < \bar{x} < \mu < \mu_2$.
(vi) $f^2(x) > x$ for $0 < x \leq \mu_1$, and $f^2(x) < x$ for $\mu \leq x < \bar{x}_2$ where $\bar{x}_2$ is the smallest fixed point of f that is greater than $\bar{x}$. If $\bar{x}_2$ does not exist, then $f(x) < x$ for all $x \geq \mu$.
(vii) All of the positive fixed points of f besides $\bar{x}$, if any, occur in the interval (μ_2, ∞).
(viii) All of the prime period-two pairs of f occur in the interval (μ_1, μ).

Case 3.3.1.2.1.1. Assume that $f(x) > x$ as $x \to \infty$.

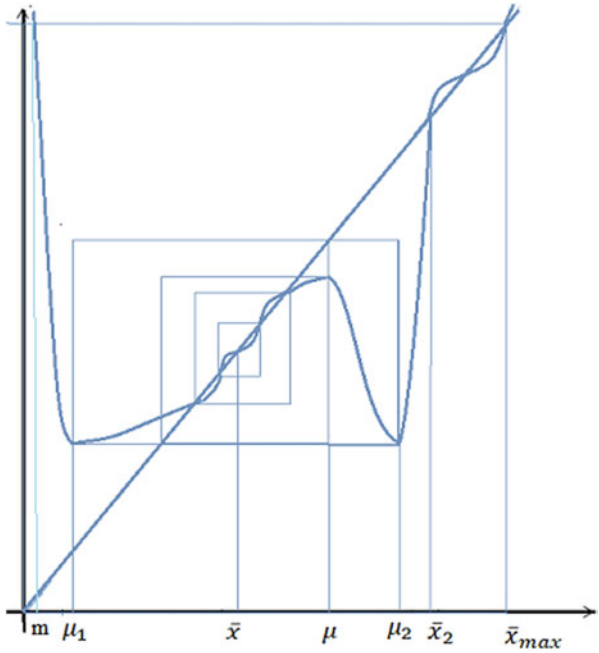

Figure 2. Graph of $y = f(f(x))$.

Since $f(\mu) < \mu$ and $\bar{x} < \mu$, then there must exist at least one other fixed point of f greater than μ. By property (vii), $\bar{x}_{\max}$ satisfies $\bar{x}_{\max} > \mu_2$. We claim that if $m > 0$ exists (defined before the statement of Theorem 3.3), then $m < \mu_1$. Since $\mu < \mu_2 < \bar{x}_{\max}$ and f is strictly increasing on $[\mu, \bar{x}_{\max}]$, then $f(\mu_2) < f(\bar{x}_{\max}) = \bar{x}_{\max}$. Since $0 < m < \mu, 0 < \mu_1 < \mu, f(\mu_1) = \mu = f(\mu_2) < f(\bar{x}_{\max}) = f(m)$, and f is strictly decreasing on $(0, \mu]$, then $m < \mu_1$ as claimed.

We are first going to prove for the case $m > 0$ that if $x_0 \in (0, m) \bigcup (\bar{x}_{\max}, \infty)$, the solution $\{f^n(x_0)\}$ diverges to infinity. Figure 2 illustrates Case 3.3.1.2.1.1 with $m > 0$. For the case $m = 0$, the result is that for $x_0 \in (\bar{x}_{\max}, \infty)$, the solution $\{f^n(x_0)\}$ diverges to infinity. We will not give the proof for the case $m = 0$, since it is a minor variation of the argument for the case $m > 0$.

Since f is strictly decreasing on $(0, m)$ and $f(m) = \bar{x}_{\max}$, then for every $x_0 \in (0, m)$, we have $f(x_0) \in (\bar{x}_{\max}, \infty)$. Next, consider the case $x_0 \in (\bar{x}_{\max}, \infty)$. But in the interval $(\bar{x}_{\max}, \infty)$, f itself is strictly increasing. Since f has no fixed points greater than $\bar{x}_{\max}$, then $f(x) > x$ for $x > \bar{x}_{\max}$. By the elementary graphic method of iteration, $f^n(x_0) \longrightarrow \infty$ as $n \longrightarrow \infty$ for $x_0 \in (\bar{x}_{\max}, \infty)$. This shows that for every $x_0 \in (0, m) \bigcup (\bar{x}_{\max}, \infty)$, the solution $\{f^n(x_0)\}$ diverges to infinity.

So, we are left to deal with $x_0 \in [m, \bar{x}_{\max}]$. Since the absolute minimum value of f is $f(\mu) = f^2(\mu_1) > \mu_1$, then $f(x_0) > \mu_1$ for $x_0 \in [m, \bar{x}_{\max}]$. In the interval $[m, \bar{x}_{\max}]$, f has absolute minima at μ_1 and μ_2, and a relative maximum at μ. Thus, the maximum value of f in this interval occurs at either m, μ, or $\bar{x}_{\max}$. Since $f^2(m) = f^2(\bar{x}_{\max}) = \bar{x}_{\max}$ and $f^2(\mu) < \mu < \bar{x}_{\max}$, then $f(x) \leq \bar{x}_{\max}$ for $x \in [m, \bar{x}_{\max}]$. Thus, $f([m, \bar{x}_{\max}]) \subseteq [m, \bar{x}_{\max}]$.

Next, we show that for every $x_0 \in [m, \bar{x}_{\max}]$, the solution $f^{2n}(x_0)$ converges to a fixed point of f^2 in this interval. Since the absolute minimum value of f^2 is $f^2(\mu_1) > \mu_1$, then we can assume $x_0 \in [\mu_1, \bar{x}_{\max}]$. Consider the case $x_0 \in [\bar{x}_2, \bar{x}_{\max}]$. Since $f^2(\bar{x}_2) =$

$\bar{x}_2$, $f^2(\bar{x}_{\max}) = \bar{x}_{\max}$ and f^2 is strictly increasing on the interval $[\bar{x}_2, \bar{x}_{\max}]$, by Theorem 2.10 the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in this interval. Next, suppose that $x_0 \in [\mu, \bar{x}_2)$. Since $f^2(x) < x$ in this interval, then for k sufficiently large we have $f^{2k}(x_0) \leq \mu$. So, we are left to consider $x_0 \in [\mu_1, \mu]$. Since $f^2(\mu_1) > \mu_1$, $f^2(\mu) < \mu$, and f^2 is strictly increasing on the interval $[\mu_1, \mu]$, then by Theorem 2.10 we have $f^{2n}(x_0)$ converges to a fixed point in this interval. So we have shown that for every $x_0 \in [m, \bar{x}_{\max}]$, $f^{2n}(x_0)$ converges to a fixed point of f^2 in this interval. Since $f([m, \bar{x}_{\max}]) \subseteq [m, \bar{x}_{\max}]$, by Lemma 2.8(c), for every $x \in [m, \bar{x}_{\max}]$, $f^n(x_0)$ converges to a fixed point of f or to a prime period-two pair of f in this interval.

We point out that if m does not exist, then the proof is similar to the proof above and even somewhat simpler, and we dispense with the details.

Case 3.3.1.2.1.2. Assume that $f(x) < x$ as $x \to \infty$.

First suppose that $x_0 \in (0, \mu_1]$. Since the absolute minimum of f^2 is $f^2(\mu_1) > \mu_1$, then $f^2(x_0) \geq \mu_1$. So we can assume that $x_0 \in [\mu_1, \infty)$. Define $\bar{x}_2$ to be the smallest fixed point of f besides $\bar{x}$, if it exists. If $\bar{x}$ does not exist, take $\bar{x} = \infty$. Consider the case $x_0 \in [\bar{x}_2, \infty)$. We showed earlier that $\bar{x}_2 > \mu_2 > \mu$. The function f itself is strictly increasing on the interval $[\bar{x}_2, \infty)$. Since $f(\bar{x}_2) = \bar{x}_2$ and $f(x) < x$ as $x \to \infty$, by Theorem 2.9, the solution $\{f^n(x_0)\}$ converges to a fixed point of f^2 in the interval $[\bar{x}_2, \infty)$.

We are left to handle the case $x_0 \in [\mu_1, \bar{x}_2)$. By the same argument as in Case 3.3.1.2.1.1, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point in this interval. We have now shown that $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 for every $x_0 > 0$. Since 0 is not a fixed point of f^2 (f^2 is strictly decreasing on $(0, \mu_1]$), then by Lemma 2.8(b), for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges either to a positive fixed point of f or to a positive prime period-two pair of f.

Case 3.3.1.2.2. There is exactly one positive solution to $f(x) = \mu$, say $x = \mu_1$, and consider the case $\mu_1 < \mu$.

This is a somewhat simpler case than Case 3.3.1.2.1 with many of the same arguments in the proof. Similar to Case 3.3.1.2.1, one can show that f^2 has the following properties:

(i) f has exactly one fixed point $\bar{x}$ and $\mu_1 < \bar{x} < \mu$.
(ii) Any prime period-two points of f must lie in the interval (μ_1, μ).
(iii) f^2 has an absolute minimum at μ_1.
(iv) f^2 is strictly decreasing on the interval $(0, \mu_1]$.
(v) f^2 is strictly increasing on the interval $[\mu_1, \mu]$.
(vi) $f^2(x) > x$ for $0 < x \leq \mu_1$ and $f^2(x) < x$ for $x \geq \mu$.

We claim that for every $x_0 > 0$, solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. First consider the case $x_0 \in (0, \mu_1]$. Since the absolute minimum value of f is $f(\mu) = f^2(\mu_1) > \mu_1$, then we are reduced to considering the case $x_0 \in (\mu_1, \infty)$. First, examine when x_0 is in the subinterval $[\mu, \infty)$. Since $f^2(x) < x$ for $x \in [\mu, \infty)$, then for k sufficiently large we have $f^{2k}(x_0) \leq \mu$. So we are left dealing with $x_0 \in [\mu_1, \mu]$. Since f^2 is strictly increasing in this interval, $f^2(\mu_1) > \mu_1$, and $f^2(\mu) < \mu$, then by Theorem 2.10 every solution $\{f^{2n}(x_0)\}$ must converge to a fixed point of f^2 in this interval. Similar to the argument in Case 3.3.1.2.1.2, for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges either to a fixed point of f or prime period-two pair of f in the interval $[\mu_1, \mu]$.

Case 3.3.1.2.3. There is exactly one positive solution to $f(x) = \mu$, say $x = \mu_2$, and $\mu < \mu_2$.

Since $f(\mu) = f^2(\mu_2)$ is the absolute minimum value of f^2 and μ is the only other critical point of f^2, then f^2 is strictly decreasing on the interval $[\mu, \mu_2]$ and strictly increasing on $[\mu_2, \infty)$. Since $f(\mu) < \bar{x} < \mu$ and f is strictly decreasing on $(0, \mu]$, then

$f^2(\mu) > f(\bar{x}) = f^2(\bar{x})$. Thus, f^2 is strictly increasing on $(0, \mu]$. Since there is no x-value where $f(x) = \mu$ for $x < \mu$, then $f(x) < \mu$ for all $0 < x < \mu$. Since f is strictly decreasing on $(0, \mu]$, then $\lim_{x \to 0} f(x) = a$ for some finite positive value of a. Thus, f can be extended to be continuous at 0. This means that f^2 is also continuous at 0, and note that $\lim_{x \to 0} f^2(x) = \lim_{x \to a} f(x) = f(a) > 0$.

We sum up the properties of f^2:

(i) In the interval $(0, \mu)$, f has exactly one fixed point $\bar{x}$.
(ii) If f has any more fixed points $\bar{x}'$ besides $\bar{x}$, then $\bar{x}' > \mu_2$.
(iii) Any prime period-two point of f must lie in the interval $(0, \mu)$.
(iv) f^2 has a relative maximum at μ and an absolute minimum at μ_2 with $f^2(\mu_2) = f(\mu)$.
(v) f^2 is strictly decreasing on the interval $[\mu, \mu_2]$.
(vi) f^2 is strictly increasing on the intervals $(0, \mu]$ and $[\mu_2, \infty)$.
(vii) $f^2(x) < x$ for $x \in [\mu, \bar{x}_2)$, where $\bar{x}_2$ is the smallest fixed point such that $\bar{x}_2 > \bar{x}$. If $\bar{x}_2$ does not exist, then $f^2(x) < x$ for all $x \in [\mu, \infty)$.
(viii) f^2 can be extended to be continuous at 0 and $f^2(0) > 0$.

Case 3.3.1.2.3.1. Suppose that $f(x) > x$ as $x \to \infty$.

By the same reasoning as in Case 3.3.1.2.1.1, there exists $\bar{x}_{\max} > \mu$ such that $\bar{x}_{\max}$ is a fixed point of f, $f(x) > x$ for all $x > \bar{x}_{\max}$, and f is strictly increasing on $[\bar{x}_{\max}, \infty)$. By Theorem 2.9, for every $x_0 \in (\bar{x}_{\max}, \infty)$, the solution $\{f^n(x_0)\}$ diverges to infinity. Clearly, $f(\bar{x}_{\max}) = \bar{x}_{\max}$. So we are left to deal with the case $x_0 \in (0, \bar{x}_{\max})$. If we can show that $\{f^n(x_0)\}$ converges to either a positive fixed point of f or a positive prime period-two pair of f for every $x_0 \in (0, \bar{x}_{\max})$, then we will be done with Case 3.3.1.2.3.1. Before we prove this, first consider the other case when $f(x) < x$ as $x \to \infty$.

Case 3.3.1.2.3.2. Suppose that $f(x) < x$ as $x \to \infty$.

If there exists a fixed point of $f > \mu$, then $\bar{x}_{\max} > \mu$ must exist. Since f is strictly increasing on $[\bar{x}_{\max}, \infty)$, then by Theorem 2.9 the solution $\{f^n x_0]$ converges to $\bar{x}_{\max}$ for every $x_0 \in [\bar{x}_{\max}, \infty)$. If $\bar{x}_{\max}$ does not exist, then choose $\bar{x}_{\max} = \infty$. We are left to show that $f^n(x_0)$ converges to either a fixed point of f or a positive prime period-two pair of f for every $x_0 \in (0, \bar{x}_{\max})$.

Thus, Cases 3.3.1.2.3.1 and 3.3.1.2.3.2 reduce to considering when $x_0 \in (0, \bar{x}_{\max})$. The first half of the proof will be the same for both cases to handle these initial values. Define $\bar{x}_2$ to be the smallest fixed point of f besides $\bar{x}$, if it exists. If $\bar{x}_2$ does not exist, choose $\bar{x}_2 = \infty$. By property (ii), $\bar{x}_2 > \mu_2 > \mu$. Suppose $x_0 \in [\bar{x}_2, \bar{x}_{\max}]$. Note that if $\bar{x}_2 = \bar{x}_{\max}$ or $\bar{x}_2 = \infty$, then there is nothing to do. Since f is strictly increasing on this interval, by Theorem 2.10, the solution $\{f^n(x_0)\}$ converges to a fixed point of f in this interval. We are left dealing with the case $x_0 \in (0, \bar{x}_2)$. Since $f^2(x) < x$ for $x \in [\mu, \bar{x}_2)$ by property (vii), then for k sufficiently large we have $f^{2k}(x_0) \leq \mu$. So, we are reduced to considering $x_0 \in (0, \mu]$. Since f^2 is strictly increasing and continuous on the interval $[0, \mu]$, $f^2(0) > 0$, and $f^2(\mu) < \mu$, then by Theorem 2.10, for every $x_0 \in (0, \mu]$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in this interval. We have shown that for every $x_0 \in (0, \bar{x}_{\max}]$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point in this interval.

Suppose $\bar{x}_{\max} < \infty$. Since the only critical points of f^2 in the interval $[0, \bar{x}_{\max}]$ are μ and μ_2, and $f^2(\mu_2)$ is the absolute minimum value of f^2, then the maximum value of f^2 occurs at either 0, μ, or $\bar{x}_{\max}$. However, f^2 is strictly increasing on $[0, \mu]$, so that we only have to consider μ or $\bar{x}_{\max}$. Note that $f^2(\mu) < \mu < \bar{x}_{\max}$. Thus, the absolute maximum value of f^2 in the interval $[0, \bar{x}_{\max}]$ is $f^2(\bar{x}_{\max}) = \bar{x}_{\max}$. Thus, $f([0, \bar{x}_{\max}]) \subseteq [0, \bar{x}_{\max}]$. Since f^2 can be extended to be continuous at 0 and $f^2(0) > 0$, then by Lemma 2.8(a), for every $x_0 \in (0, \bar{x}_{\max}]$, the solution $\{f^n(x_0)\}$ converges either to a positive fixed point of f or to a positive prime period-two pair of f. In the event that $\bar{x}_{\max} = \infty$, we can invoke Lemma 2.8(b) to obtain the result that for every $x_0 \in (0, \infty)$, the solution $\{f^n(x_0)\}$ converges either to a positive fixed point of f or to a positive prime period-two pair of f.

Case 3.3.1.2.4. There are no positive solutions to $f(x) = \mu$.

Since f is continuous and $f(\mu) < \mu$, then $f(x) < \mu$ for all $x > 0$. Suppose $0 < x_1 < x_2 < \mu$. Since f is strictly decreasing in the interval $(0, \mu]$, we have $\mu > f(x_1) > f(x_2)$, and so $f^2(x_1) < f^2(x_2)$. Thus, f^2 is strictly increasing on the interval $(0, \mu]$. With a similar argument, one can show that f^2 is strictly decreasing on the interval $[\mu, \infty)$. This implies that f^2 has an absolute maximum at μ, and since the absolute maximum value of f is less than μ, then $f^2(\mu) < \mu$. Since f^2 is strictly decreasing in the interval $[\mu, \infty)$ and $f^2(\mu) < \mu$, then $f^2(x) < x$ for all $x \in [\mu, \infty)$.

We claim that every solution converges to a fixed point of f^2. First consider the case $x_0 \in [\mu, \infty)$. We have $f^2(x_0) < \mu$. Thus, we are left with the case $x_0 \in (0, \mu]$. By the same argument as in Case 3.3.1.2.3, one shows that f^2 can be extended to be continuous at 0 and $f^2(0) > 0$. Since f^2 is strictly increasing in the interval $[0, \mu]$, $f^2(0) > 0$ and $f^2(\mu) < \mu$, then by Theorem 2.10 every solution $\{f^{2n}(x_0\})$ must converge to a positive fixed point of f^2.

By Lemma 2.8(b), for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges either to a positive fixed point of f or to a positive prime period-two pair of f.

Case 3.3.2. The unique relative extreme point of f, at say $x = \mu$, is a relative maximum with $\mu > 0$.

Since f has exactly one critical point at $x = \mu$ and it is a relative maximum, then f is strictly increasing for $x \leq \mu$ and strictly decreasing for $x \geq \mu$. This implies that f has an absolute maximum at $x = \mu$. Since f is strictly increasing and continuous on $(0, \mu]$, then $\lim_{x \to 0} f(x) \geq 0$ exists and is finite. Thus, we can extend f to be continuous on $[0, \infty)$.

Suppose first that $f(\mu) < \mu$. Note that $f(x) < \mu$ for all $x > 0$. Suppose $x_0 \in [\mu, \infty)$. We have $f(x_0) < \mu$. So, we are left with the case $x_0 \in (0, \mu]$. Since f is strictly increasing in the interval $[0, \mu]$, $f(0) \geq 0$ and $f(\mu) < \mu$, then by Theorem 2.10 every solution $\{f^n(x_0)\}$ must converge to a fixed point of f in the interval $[0, \mu]$.

So, we now need to consider the case when $f(\mu) > \mu$, which is the main case.

Since f is strictly decreasing on the interval $[\mu, \infty)$ and $f(\mu) > \mu$, then there is exactly one fixed point of f, say $\bar{x}$ in the interval (μ, ∞). Suppose that μ_2 exists (defined just before Case 3.3.1). We claim that $\bar{x} < \mu_2$. We have $f(\bar{x}) = \bar{x} > \mu = f(\mu_2)$. Since f is strictly decreasing on the interval (μ, ∞), then $\bar{x} < \mu_2$ as claimed.

We also claim that any prime period-two point of f must lie in the interval (μ, μ_2). By assumption any prime period-two pair of f, $(\bar{p}_1, \bar{p}_2)$ and we choose $\bar{p}_1 < \bar{p}_2$, must satisfy $\mu < \bar{p}_1 < \bar{p}_2$. Since f is strictly decreasing on $[\mu, \infty)$, $\mu < \mu_2$ and $f(\mu_2) = \mu$, then $f(x) < \mu$ for $x \in [\mu_2, \infty)$. This implies $\bar{p}_2 < \mu_2$ since $f(\bar{p}_2) = \bar{p}_1 > \mu$. Thus, $\mu < \bar{p}_1 < \bar{p}_2 < \mu_2$ as claimed.

Suppose μ_1 exists (defined just before Case 3.3.1). If f has any other fixed points besides $\bar{x}$, we claim that they must lie in the interval $[0, \mu_1)$. Let $\bar{x}' < \mu$ denote any other fixed point of f, if it exists. We have $f(\bar{x}') = \bar{x}' < \mu = f(\mu_1)$. Since f is strictly increasing on the interval $[0, \mu)$, then we must have $\bar{x}' < \mu_1$ as claimed.

Since f can be extended to be continuous at 0, by Lemma 2.8(e), we conclude that for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ converges to either a non-negative fixed point of f or a positive prime period-two pair of f if and only if $f^{2n}(x_0)$ converges to a fixed point of f^2. So, we only need to consider the convergence properties of f^2. We also point out that since

f can be extended to be continuous on $[0, \infty)$, then f^2 can also be extended to be continuous on $[0, \infty)$.

Case 3.3.2.1. There are exactly two values $\mu_1 < \mu < \mu_2$ such that $f(\mu_1) = f(\mu_2) = \mu$.

Since $f(\mu)$ is an absolute maximum of f, then the absolute maxima of f^2 occur at μ_1 and μ_2. Since $\mu_1 < \mu < \mu_2$ and these are the only critical values, then the critical value $x = \mu$ must give a relative minimum for f^2. Since all of the fixed points of f^2 are less than μ_2 and f^2 is bounded, then $f^2(x) < x$ for $x \in [\mu_2, \infty)$. Note that $f^2(\mu_1) = f(\mu) > \mu > \mu_1$. Since $f^2(\mu_1) > \mu_1$ and f^2 has no fixed points in the interval $[\mu_1, \mu]$, then $f^2(x) > x$ for $x \in [\mu_1, \mu]$.

Let us summarize the properties of f^2:

(i) We have the relationships $\mu_1 < \mu < \bar{x} < \mu_2$.
(ii) f^2 is strictly increasing on the intervals $[0, \mu_1]$ and $[\mu, \mu_2]$, and strictly decreasing on the intervals $[\mu_1, \mu]$ and $[\mu_2, \infty)$.
(iii) f has exactly one fixed point $\bar{x}$ that satisfies $\mu < \bar{x} < \mu_2$.
(iv) Any other fixed point of f besides $\bar{x}$ lies in the interval $[0, \mu_1)$.
(v) Any prime period-two point of f lies in the interval (μ, μ_2).
(vi) f^2 has absolute maxima at μ_1 and μ_2, and a relative minimum at μ.
(vii) $f^2(x) > x$ for $x \in [\mu_1, \mu]$ and $f^2(x) < x$ for $x \in [\mu_2, \infty)$.
(viii) f^2 can be extended to be continuous on $[0, \infty)$.

We claim that for every $x_0 > 0$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. First, suppose $x_0 \in [\mu_2, \infty)$. Since $f^2(\mu_2) < \mu_2$ is the absolute maximum value of f^2, then $f^2(x_0) \in [0, \mu_2]$. So we are left to deal with $x_0 \in [0, \mu_2]$. Let $\bar{x}_N$ denote the largest fixed point of f besides $\bar{x}$ if it exists. Consider the case $x_0 \in (0, \bar{x}_N]$ and we point out that $\bar{x}_N < \mu_1$ by property (iv). Since f^2 is strictly increasing on the interval $[0, \bar{x}_N], f^2(0) \geq 0$ and $f^2(\bar{x}_N) = \bar{x}_N$, then by Theorem 2.10 every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in the interval $[0, \bar{x}_N]$. Next, consider the case $x_0 \in (\bar{x}_N, \mu)$ and take $\bar{x}_N = 0$ if f has no other fixed point besides $\bar{x}$. Since there are no fixed points of f^2 on the interval $(\bar{x}_N, \mu]$ and $f^2(\mu) > \mu$, then $f^2(x) > x$ for $x \in (\bar{x}_N, \mu]$. Thus, there exists k sufficiently large so that $f^{2k}(x_0) \geq \mu$. Since $f^2(\mu_2) < \mu_2$ is the absolute maximum value of f^2, then $\mu \leq f^{2k}(x_0) \leq \mu_2$. So, we are left to consider the case $x_0 \in [\mu, \mu_2]$. Since f^2 is strictly increasing on this interval, $f^2(\mu) \geq \mu$ and $f^2(\mu_2) < \mu_2$, then by Theorem 2.10, every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in this interval.

Case 3.3.2.2. Assume there is a unique value μ_1 such that $f(\mu_1) = \mu$ and consider the case $\mu_1 < \mu$.

Since $f(\mu)$ is the absolute maximum of f, then the unique absolute maximum of f^2 occurs at μ_1. Observe that $(f^2)'(\bar{x}) = f'(f(\bar{x}))f'(\bar{x}) = [f'(\bar{x})]^2 > 0$. Since μ_1 and μ are the only critical values of f^2, f^2 has an absolute maximum at $\mu_1, (f^2)'(\bar{x}) > 0$ and $\mu_1 < \mu < \bar{x}$, we conclude that f^2 must have a relative minimum at μ. Thus, f^2 is strictly increasing on the intervals $[0, \mu_1]$ and $[\mu, \infty)$, and strictly decreasing on the interval $[\mu_1, \mu]$. Let $\bar{x}_N$ denote the largest fixed point of f besides $\bar{x}$. If $\bar{x}$ is the only fixed point of f, then take $\bar{x}_N = 0$.

By the same argument as in Case 3.3.2.1, we have $f^2(x) > x$ for $x \in (\bar{x}_N, \mu]$. Since the absolute maximum value of f^2 is $f^2(\mu_1) < \infty$, then $f^2(x) < x$ as $x \to \infty$.

Let us summarize the properties of f^2:

(i) f^2 is strictly increasing on the intervals $[0, \mu_1]$ and $[\mu, \infty)$, and strictly decreasing on the interval $[\mu_1, \mu]$.
(ii) f has exactly one fixed point $\bar{x}$ that satisfies $\bar{x} > \mu$.

(iii) Any other fixed point of f lies in the interval $[0, \mu_1)$.
(iv) Any prime period-two point of f lies in the interval (μ, ∞).
(v) $f^2(x) > x$ for $x \in (\bar{x}_N, \mu]$ and $f^2(x) < x$ as $x \to \infty$.
(vi) f^2 can be extended to be continuous on $[0, \infty)$.

We claim that for every $x_0 > 0$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. First, consider the case $x_0 \in (0, \bar{x}_N]$. By properties (i) and (iii), f^2 is strictly increasing in this interval. We also have $f^2(0) \geq 0$ and $f^2(\bar{x}_N) = \bar{x}_N$. By Theorem 2.10, every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in the interval $[0, \bar{x}_N]$. Next, consider the case $x_0 \in (\bar{x}_N, \mu]$. Since $f^2(x) > x$ for $x \in (\bar{x}_N, \mu]$, then for k sufficiently large we have $f^{2k}(x_0) \geq \mu$. Thus, we are only left to consider $x_0 \in [\mu, \infty)$. Since f^2 is strictly increasing in this interval, $f^2(\mu) > \mu$ and $f^2(x) < x$ as $x \to \infty$, then by Theorem 2.9 every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in this interval.

Case 3.3.2.3. Assume there is a unique value μ_2 such that $f(\mu_2) = \mu$ and consider the case $\mu_2 > \mu$.

Since $f(\mu)$ is the absolute maximum of f, then the unique maximum of f^2 occurs at μ_2. Since $f(\mu) > \mu$ and there is no value of x such that $f(x) = \mu$ with $0 \leq x < \mu$, then $f(x) > \mu$ for all $x \in [0, \mu]$. Thus, f has no fixed point in the interval $[0, \mu]$. By assumption, there are no prime period-two points of f in the interval $[0, \mu]$. Since there are no fixed points for f^2 in the interval $[0, \mu]$ and $f(0) > \mu$, then $f^2(x) > x$ for $x \in [0, \mu]$. Since $f^2(\mu_2)$ is the unique absolute maximum value of f^2, and μ and μ_2 are the only critical values of f^2, then f^2 is strictly increasing in the interval $[\mu, \mu_2]$. It also follows that since $f^2(\mu_2)$ is the absolute maximum of f^2 and there are no critical values greater than μ_2, then f^2 is strictly decreasing on the interval $[\mu_2, \infty)$. Since f^2 is bounded and has no fixed point in the interval $[\mu_2, \infty)$, then $f^2(x) < x$ for $x \in [\mu_2, \infty)$.

Let us summarize the properties of f^2:

(i) f^2 is strictly increasing on the interval $[\mu, \mu_2]$ and strictly decreasing on the interval $[\mu_2, \infty)$.
(ii) f has exactly one fixed point $\bar{x}$ and it satisfies $\mu < \bar{x} < \mu_2$.
(iii) Any prime period-two point of f lies in the interval (μ, μ_2).
(iv) $f^2(x) > x$ for $x \in [0, \mu]$ and $f^2(x) < x$ for $x \in [\mu_2, \infty)$.

We claim that for every $x_0 > 0$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. First, suppose $x_0 \in [\mu_2, \infty)$. Since $f^2(\mu_2) < \mu_2$ is the absolute maximum of f^2, then $f^2(x_0) \in [0, \mu_2]$. Next suppose $x_0 \in [0, \mu]$. Since $f^2(x) > x$ for $x \in [0, \mu]$ and the absolute maximum value of f^2 is less than μ_2, then for k sufficiently large we have $f^{2k}(x_0) \in [\mu, \mu_2]$. We are left to consider the case $x_0 \in [\mu, \mu_2]$. Since f^2 is strictly increasing in this interval, $f^2(\mu) > \mu$ and $f^2(\mu_2) < \mu_2$, then by Theorem 2.10 every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in this interval.

Case 3.3.2.4. Assume there is no value x such that $f(x) = \mu$.

Similar to the arguments in Case 3.3.1.2.4, we have f^2 is strictly decreasing in the interval $[0, \mu]$ and strictly increasing in the interval $[\mu, \infty)$. Since there is no x-value such that $f(x) = \mu$ and $f(\mu) > \mu$, then for all $x \in [0, \mu]$ we have $f(x) > \mu$. Thus, there are no fixed points of f in the interval $[0, \mu]$. Since f is strictly decreasing on (μ, ∞) and $f(\mu) > \mu$, then there must exist exactly one fixed point $\bar{x}$ of f in this interval. We conclude that f has exactly one fixed point in $[0, \infty)$. By assumption, f has no prime period-two points in the interval $[0, \mu]$. Since f^2 is strictly decreasing on $[0, \mu]$, and has no fixed point in this interval, then $f^2(x) > x$ for $x \in [0, \mu]$. From the fact that the absolute maximum value of f^2 is $f(\mu)$, we conclude that $f^2(x) < x$ as $x \to \infty$.

Let us summarize the properties of f^2:

(i) f^2 is strictly decreasing on the interval $[0, \mu]$ and strictly increasing on the interval $[\mu, \infty)$.
(ii) f has exactly one fixed point $\bar{x}$ and it satisfies $\bar{x} > \mu$.
(iii) Any prime period-two point of f lies in the interval (μ, ∞).
(iv) $f^2(x) > x$ for $x \in [0, \mu]$ and $f^2(x) < x$ as $x \to \infty$.

We claim that for every $x_0 > 0$, the solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2. First, suppose $x_0 \in (0, \mu)$. Since $f^2(x) > x$ for $x \in [0, \mu]$, then for k sufficiently large we have $f^{2k}(x_0) \geq \mu$. We are left to deal with the case $x_0 \in [\mu, \infty)$. Since f^2 is strictly increasing in this interval, $f^2(\mu) > \mu$ and $f^2(x) < x$ as $x \to \infty$, then by Theorem 2.9, every solution $\{f^{2n}(x_0)\}$ converges to a fixed point of f^2 in the interval $[\mu, \infty)$. □

4. Applications to rational difference equations

In this section, we consider all difference equations of the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_n + C}{\alpha x_n^2 + \beta x_n + \gamma} \tag{2}$$

with non-negative parameters A, B, C, α, β and γ. We will find necessary and sufficient conditions on the parameters and positive initial values for convergence of solutions of this equation. Any case that reduces to the form $x_{n+1} = (Dx_n + E)/(\delta x_n + \epsilon)$ will not be addressed, since the convergence properties of this difference equation are known. All cases of this type are covered in a book by Camouzis and Ladas [3] in Appendix A. The relevant difference equations are numbered (17), (23), (41), (42) and (65) in Appendix A. We will also not cover the case when $x_{n+1} = Ax_n^2 + Bx_n + C$, as the convergence properties of this case are well known.

THEOREM 4. *The solutions of the difference equation (2) above converge either to a fixed point of f or to a prime period-two pair of f for every non-negative choice of the parameters and every choice of positive initial values, with the following exceptions:*

(a) *Suppose $A = 0, B > 0, C > 0, \alpha > 0, \beta < \alpha C/B$ and $\gamma = 0$. Let $\bar{x}$ denote the unique positive fixed point of f that is a root of the equation $\alpha\bar{x}^3 + \beta\bar{x}^2 - B\bar{x} - C = 0$. Then for every $x_0 \in (0, \bar{x})$ the solutions $f^{2n}(x_0) \to 0$ and $f^{2n+1}(x_0) \to \infty$ as $n \to \infty$. For every $x_0 \in (\bar{x}, \infty)$ the solutions $f^{2n}(x_0) \to \infty$ and $f^{2n+1}(x_0) \to 0$ as $n \to \infty$.*
(b) *Suppose $A = 0$, $B = 0$, $C > 0$, $\alpha > 0$ and $\gamma = 0$. Let $\bar{x}$ denote the unique positive fixed point of f which is a root of $\alpha\bar{x}^3 + \beta\bar{x}^2 - C = 0$. Then for every $x_0 \in (0, \bar{x})$ the solutions $f^{2n}(x_0) \to 0$ and $f^{2n+1}(x_0) \to \infty$ as $n \to \infty$. For every $x_0 \in (\bar{x}, \infty)$ the solutions $f^{2n}(x_0) \to \infty$ and $f^{2n+1}(x_0) \to 0$ as $n \to \infty$.*
(c) *Suppose that $A > \beta > 0$ and $\alpha = 0$. Also assume that either $(\gamma - B)^2 < 4(A - \beta)C$ or $B \geq \gamma$. In this case, for every $x_0 > 0$, the solution $f^n(x_0) \to \infty$ as $n \to \infty$.*
(d) *Suppose that $A > \beta > 0, \alpha = 0, (\gamma - B)^2 \geq 4(A - \beta)C$ and $B < \gamma$.*

Let $\bar{x}_{\max}$ denote the largest fixed point of f which equals

$$\bar{x}_{\max} = \frac{\gamma - B + \sqrt{(\gamma - B)^2 - 4(A - \beta)C}}{2(A - \beta)}.$$

If $x_0 \in (0, \bar{x}_{\max}]$, *then the solution* $\{f^n(x_0)\}$ *converges to a fixed point of* f. *If* $x_0 \in (\bar{x}_{\max}, \infty)$ *then the solution* $f^n(x_0) \to \infty$ *as* $n \to \infty$.

(e) *Suppose that* $A = \beta > 0, \alpha = 0$, *and either* $B > \gamma$ *or both* $B = \gamma$ *and* $C > 0$. *Then for every* $x_0 > 0$, *the solution* $f^n(x_0) \to \infty$ *as* $n \to \infty$.

Proof.

Case 4.1. Assume that $A > 0$ and $\alpha > 0$.

By the change of variables $x_n \to Ax_n/\alpha$, we can assume $A = \alpha = 1$. Equation (2) can be expressed in the form:

$$x_{n+1} = \frac{x_n^2 + Bx_n + C}{x_n^2 + \beta x_n + \gamma} \equiv f(x_n). \tag{3}$$

We can also assume that C and γ are not both zero, since otherwise the difference equation reduces to the well-known linear case in numerator and denominator.

We calculate the derivative of $f(x)$:

$$f'(x) = \frac{(\beta - B)x^2 + 2(\gamma - C)x + B\gamma - C\beta}{(x^2 + \beta x + \gamma)^2}. \tag{4}$$

Setting the derivative of $f(x)$ equal to zero, we find that the critical values must satisfy

$$(\beta - B)x^2 + 2(\gamma - C)x + B\gamma - C\beta = 0. \tag{5}$$

We claim that f has at most one positive critical value, as we now show breaking the argument up into Cases a, b and c.

Case a. Suppose $\beta - B > 0$.

A necessary condition for two positive solutions to equation (5) is $\gamma - C < 0$ and $B\gamma - C\beta > 0$. However, $\beta > B$ and $\gamma < C$ imply $B\gamma < C\beta$.

Case b. Suppose $\beta - B < 0$.

A necessary condition for two positive solutions to equation (5) is $\gamma - C > 0$ and $B\gamma - C\beta < 0$. However, $\beta < B$ and $\gamma > C$ imply $B\gamma > C\beta$.

Case c. Suppose $\beta - B = 0$.

If $\beta - B = 0$, then equation (5) clearly has at most one root.

We next claim that if f has no relative extreme point, then every solution converges either to a fixed point or to a positive prime period-two pair of f. For if f has no relative extreme point, then f must be strictly increasing or strictly decreasing for all $x > 0$. First consider the case that f is strictly increasing. Inspecting equation (5), we obviously must have $\gamma > 0$. This then implies that f is bounded and so $f(x) < x$ as $x \to \infty$. Since f is strictly increasing, we can extend f so that it is continuous at the origin and $f(0) \geq 0$. By Theorem 2.9, for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges to a fixed point of f. Next, consider the case f is strictly decreasing. Observe that $\lim_{x\to\infty} f(x) = 1$. By Theorem 3.1, the solution $\{f^n(x_0)\}$ converges to either the unique fixed point of f or a positive prime period-two pair of f.

We now consider the main case that there is exactly one relative extreme point for f. We first investigate the number of fixed points of f. It is easy to check that any fixed point

$\bar{x}$ of f must be a root of the equation

$$g(\bar{x}) \equiv \bar{x}^3 + (\beta - 1)\bar{x}^2 + (\gamma - B)\bar{x} - C = 0. \tag{6}$$

If $C = 0$, then clearly $\bar{x} = 0$ is a fixed point. If $C > 0$, then there is certainly at least one positive fixed point since $g(0) = -C$ and $g(x) > 0$ for x sufficiently large. Since $g(\bar{x})$ is a cubic, then there are at most three fixed points. Recall that $f(x) < x$ as $x \to \infty$. If f has no prime period-two pair, then by Theorem 3.3, for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ converges to a fixed point of f.

So, we are left to deal with the situation that f has exactly one critical point that gives a relative extreme point, and f has at least one prime period-two pair.

From equation (3), we see that any prime period-two pair must satisfy both of the following two equations:

$$\bar{p}_2 = \frac{\bar{p}_1^2 + B\bar{p}_1 + C}{\bar{p}_1^2 + \beta\bar{p}_1 + \gamma}$$

and

$$\bar{p}_1 = \frac{\bar{p}_2^2 + B\bar{p}_2 + C}{\bar{p}_2^2 + \beta\bar{p}_2 + \gamma}.$$

After some algebraic manipulation, one can show that

$$\bar{p}_1 + \bar{p}_2 = \frac{C - (B + \gamma)(\beta + 1)}{\gamma + \beta + 1}. \tag{7}$$

Observe that since $f \in C((0, \infty) \times (0, \infty))$, then by Lemma 2.7, any prime period-two pair of f must be a positive prime period-two pair. If there is a positive prime period-two pair, obviously it must satisfy

$$\max(\bar{p}_1, \bar{p}_2) < \frac{C - (B + \gamma)(\beta + 1)}{\gamma + \beta + 1}. \tag{8}$$

This shows that a necessary condition to obtain a positive prime period-two pair is that

$$C > (B + \gamma)(\beta + 1). \tag{9}$$

In the remainder of the proof of Theorem 4, denote the unique positive critical value of f by μ.

Case 4.1.1. Assume that $\beta > B$.

Since we are dealing with the case that there is at least one prime period-two pair, then from equation (9) we see that $C > \gamma$. Observe that $B\gamma < C\beta$. From equation (4), we see that $f'(x) < 0$ for x sufficiently small and $f'(x) > 0$ for x sufficiently large. Thus, the critical value at μ gives a relative minimum for f. Observe that $\lim_{x\to\infty} f(x) = 1$ so that $f(x) < x$ as $x \to \infty$. We will be able to use Theorem 3.3 to conclude that every solution $f^n(x_0)$ will converge to a positive fixed point or positive prime period-two pair of f as long as we show that every prime period-two point of f is less than μ.

Let $\mu' \leq 0$ be the other root of equation (5). Then we have

$$\mu + \mu' = \frac{2(C - \gamma)}{\beta - B}. \tag{10}$$

This implies

$$\mu \geq \frac{2(C-\gamma)}{\beta - B}. \tag{11}$$

By equations (8) and (11), it is sufficient to show

$$\frac{C-(B+\gamma)(\beta+1)}{\gamma+\beta+1} \leq \frac{2(C-\gamma)}{\beta-B}. \tag{12}$$

Thus, we only need to show that

$$[C-(B+\gamma)(\beta+1)](\beta-B) \leq 2(C-\gamma)(\gamma+\beta+1).$$

This is equivalent to showing

$$C(2\gamma+\beta+B+2) \geq 2\gamma(\gamma+\beta+1)+(B-\beta)(B+\gamma)(\beta+1).$$

Since $C > (B+\gamma)(\beta+1)$, it is sufficient if

$$(B+\gamma)(\beta+1)(2\gamma+\beta+B+2) \geq 2\gamma(\gamma+\beta+1)+(B-\beta)(B+\gamma)(\beta+1),$$

which reduces to

$$2(\gamma+\beta+1)[\beta(\gamma+B)+B] \geq 0,$$

and this inequality is clearly true.

Case 4.1.2. Assume that $\beta < B$.

Recall that equation (9) implied that $\gamma - C < 0$. From equation (5), we see that we must have $B\gamma - C\beta > 0$ in order for there to be a relative extreme point. This implies that $\gamma > 0$. In equation (3), make the change of variables $y_n = C/(\gamma x_n)$. Equation (3) can be rewritten in the form

$$y_{n+1} = h(y_n) \equiv \frac{y_n^2 + C\beta y_n/\gamma^2 + C^2/\gamma^3}{y_n^2 + By_n/\gamma + C/\gamma^2}. \tag{13}$$

Since both $C > 0$ and $\gamma > 0$, observe that 0 is not a fixed point of h. Thus, $\bar{x} > 0$ is a fixed point of f if and only if $C/(\gamma\bar{x})$ is a positive fixed point of h. Similarly, $(\bar{p}_1, \bar{p}_2)$ is a positive prime period-two pair of f if and only if $(C/(\gamma\bar{p}_1), C/(\gamma\bar{p}_2))$ is a positive prime period-two pair of h. If h is a monotone function, then by Theorem 2.9 or Theorem 3.1, for every $y_0 > 0$, the solution $\{h^n(y_0)\}$ converges to either a positive fixed point or positive prime period-two pair of h. By continuity, choosing $y_0 = C/(\gamma x_0)$, for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ converges to either a positive fixed point of f or positive prime period-two pair of f. If h is not monotone, then it must have exactly one critical value that is a relative extreme point. Since $B\gamma > C\beta$, then $B/\gamma > C\beta/\gamma^2$. Thus, we are back in Case 4.1.1. By that case, for every $y_0 > 0$, the solution $\{h^n(y_0)\}$ must converge to either a positive fixed point or positive prime period-two pair of h. Once again by continuity, for every $x_0 > 0$, the solution $\{f^n(x_0)\}$ converges to either a positive fixed point of f or positive prime period-two pair of f.

Case 4.2. Assume that $A = 0, B > 0$, and $\alpha > 0$.

After the change of variables $x_n \to x_n\sqrt{B/\alpha}$, we can assume that $B = 1$ and $\alpha = 1$. So, the difference equation has the form

$$x_{n+1} = \frac{x_n + C}{x_n^2 + \beta x_n + \gamma} \equiv f(x_n). \tag{14}$$

It is easy to check that any prime period-two pair $(\bar{p}_1, \bar{p}_2)$ are roots of the equation

$$\gamma x^2 + [\beta(1+\gamma) - C]x + \gamma(1+\gamma) = 0. \tag{15}$$

If $\gamma > 0$, in order for these two roots to be positive, it is necessary and sufficient that

$$C > \beta(1+\gamma) \tag{16}$$

and

$$[C - \beta(1+\gamma)]^2 > 4\gamma^2(1+\gamma). \tag{17}$$

If $\gamma = 0$, then there are no prime period-two pairs. Since

$$f'(x) = \frac{-x^2 - 2Cx + \gamma - \beta C}{(x^2 + \beta x + \gamma)^2}, \tag{18}$$

any critical value must be a root of the equation

$$-x^2 - 2Cx + \gamma - \beta C = 0. \tag{19}$$

Also note that $f(x) < x$ as $x \to \infty$.

Case 4.2.1. Assume that $\gamma - \beta C \le 0$.

In this case, f is strictly decreasing. If $\gamma > 0$, then $\lim_{x\to 0} f(x) = C/\gamma < \infty$. By Theorem 3.1, for every $x_0 > 0$ the solution $f^n(x_0)$ converges either to a fixed point of f or to a positive prime period-two pair of f. If $\gamma = 0$ and $C = 0$, the difference equation reduces to the well-known case $f(x) = 1/(x+\beta)$. If $\gamma = 0$ and $C > 0$, then a straightforward calculation shows that $\lim_{x\to 0} f^2(x)/x = \beta/C$. If $\gamma = 0$ and $\beta > C > 0$, then by Theorem 3.1 for every $x_0 > 0$ the solution $f^n(x_0)$ converges to either a fixed point or a positive prime period-two pair of f. If $\gamma = 0$ and $\beta = C > 0$, then $f(x) = 1/x$ which is well known. We saw earlier that f has no prime period-two pair when $\gamma = 0$. Let $\bar{x}$ denote the unique positive fixed point of f. If $\gamma = 0, C > 0$ and $\beta < C$, by Theorem 3.1, for $0 < x_0 < \bar{x}$ we have $f^{2n}(x_0) \to 0$ and $f^{2n+1}(x_0) \to \infty$ as $n \to \infty$. When $\bar{x} < x_0 < \infty$ we have $f^{2n}(x_0) \to \infty$ and $f^{2n+1}(x_0) \to 0$ as $n \to \infty$. Obviously if $x_0 = \bar{x}$, then that is just a fixed point of f. Recall that we made the change of variables $x_n \to x_n\sqrt{B/\alpha}$ at the beginning of Case 4.2. If we convert back to the original parameters, we obtain the following conditions: $A = 0, B > 0, C > 0, \alpha > 0, \beta < C\alpha/B$ and $\gamma = 0$. Observe that this result is exception (a) in the conclusion of Theorem 4.

Case 4.2.2. Assume that $\gamma - \beta C > 0$.

In this case, f has one unique relative extreme point that is a relative maximum, as can be seen from equation (18). It is easy to calculate that any fixed point $\bar{x}$ of f must satisfy

$$\bar{x}^3 + \beta\bar{x}^2 + (\gamma - 1)\bar{x} - C = 0. \tag{20}$$

Observe that equation (20) must have at least one non-negative root. If f does not have a positive prime period-two pair then by Theorem 3.3, for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ must converge to one of its non-negative fixed points. Suppose f does have a positive prime period-two pair. From equation (15), it is clear that it has at most one such pair of solutions, say $(\bar{p}_1, \bar{p}_2)$. From equation (15), we see that

$$\min(\bar{p}_1, \bar{p}_2) = \frac{C - \beta(1+\gamma) - \sqrt{[C-\beta(1+\gamma)]^2 - 4\gamma^2(1+\gamma)}}{2\gamma}$$

$$= \frac{C - \beta(1+\gamma)}{2\gamma}\left[1 - \sqrt{1 - \frac{4\gamma^2(1+\gamma)}{[C-\beta(1+\gamma)]^2}}\right]. \tag{21}$$

From equation (19), we see that the critical value μ must satisfy

$$\mu = -C + \sqrt{C^2 + \gamma - \beta C} = C\left[\sqrt{1 + \frac{\gamma - \beta C}{C^2}} - 1\right]. \tag{22}$$

Since $1 - \sqrt{1-a} \geq a/2$ for all $0 \leq a \leq 1$ and $\sqrt{1+a} - 1 < a/2$ for all $a > 0$, then by equations (21) and (22) we obtain

$$\min(\bar{p}_1, \bar{p}_2) \geq \frac{\gamma(1+\gamma)}{C - \beta(1+\gamma)} \tag{23}$$

and

$$\mu < \frac{\gamma - \beta C}{2C}. \tag{24}$$

By Theorem 3.3, every solution of f will converge either to a fixed point of f or to the unique positive prime period-two pair as long as $\mu \leq \min(\bar{p}_1, \bar{p}_2)$. By equations (23) and (24), it is sufficient to show that

$$\frac{\gamma - \beta C}{2C} \leq \frac{\gamma(1+\gamma)}{C - \beta(1+\gamma)},$$

or equivalently,

$$C(\gamma + 2\gamma^2 + \beta C) + \beta(1+\gamma)(\gamma - \beta C) \geq 0.$$

However, this equation is true since $\gamma - \beta C > 0$.

Case 4.3. Assume that $A = 0$, $B = 0$, $C > 0$ and $\alpha > 0$.

The function f has the form

$$f(x) = \frac{C}{\alpha x^2 + \beta x + \gamma}. \tag{25}$$

It is easy to check that f is strictly decreasing.

Case 4.3.1. Assume that $\gamma > 0$.

In this case, we have $\lim_{x\to 0} f(x) = C/\gamma$. By Theorem 3.1, every solution of f converges to either a positive fixed point of f or a positive prime period-two pair of f.

Case 4.3.2. Assume that $\gamma = 0$.

It is easy to calculate that $\lim_{x\to 0} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to 0} f^2(x)/x^2 = \beta^2/(\alpha C)$. It is also easy to check that there are no positive prime period-two pairs.

Since f is strictly decreasing, there must be a unique positive fixed point, say $\bar{x}$, of f which is a root of $\alpha\bar{x}^3 + \beta\bar{x}^2 - C = 0$. By Theorem 3.1, for every $x_0 \in (0, \bar{x})$, $f^{2n}(x_0)$ converges to 0 and $f^{2n+1}(x_0)$ diverges to infinity. Furthermore, for every $x_0 \in (\bar{x}, \infty)$, $f^{2n}(x_0)$ diverges to infinity and $f^{2n+1}(x_0)$ converges to 0. Observe that this case is exception (b) in the conclusion of Theorem 4.

Case 4.4. Assume that $A > 0$, $\alpha = 0$ and $\beta > 0$.

The difference equation takes the form

$$x_{n+1} = \frac{Ax_n^2 + Bx_n + C}{\beta x_n + \gamma} \equiv f(x_n). \tag{26}$$

In this case, there are no positive prime period-two pairs since a simple calculation shows that any positive prime period-two pair $(\bar{p}_1, \bar{p}_2)$ must satisfy $A(\bar{p}_1 + \bar{p}_2) = -\gamma - B$. It is easy to check that any fixed point $\bar{x}$ of f must satisfy

$$(A - \beta)\bar{x}^2 + (B - \gamma)\bar{x} + C = 0. \tag{27}$$

It is easy to calculate that

$$f'(x) = \frac{A\beta x^2 + 2A\gamma x + B\gamma - C\beta}{(\beta x + \gamma)^2}. \tag{28}$$

Thus, f has at most one positive critical value.

Observe that if $B\gamma - C\beta \geq 0$, then f is strictly increasing. If $B\gamma - C\beta < 0$, then f has a unique positive relative minimum.

Case 4.4.1. Assume that $A > \beta$.

Case 4.4.1.1. Assume that either $(\gamma - B)^2 < 4(A - \beta)C$ or $B - \gamma \geq 0$.

In this case, equation (27) has no positive roots and thus there are no positive fixed points for f. Since $f(x)/x \to A/\beta > 1$ as $x \to \infty$, then clearly $f(x) > x$ for all $x > 0$. This implies that $f^n(x_0) \to \infty$ as $n \to \infty$ for every $x_0 > 0$. Observe that this case is exception (c) in the conclusion of Theorem 4.

Case 4.4.1.2. Assume that $(\gamma - B)^2 \geq 4(A - \beta)C$ and $B - \gamma < 0$.

In this case, there are one or two non-negative fixed points. The larger (or only) fixed point of f is

$$\bar{x}_{\max} = \frac{\gamma - B + \sqrt{(\gamma - B)^2 - 4(A - \beta)C}}{2(A - \beta)}.$$

Observe that $f(x)/x \to A/\beta > 1$ as $x \to \infty$.

Case 4.4.1.2.1. Assume that $B\gamma - C\beta \geq 0$.

As we observed earlier by examining equation (28), in this case f is strictly increasing. By Theorem 2.9, the solution $\{f^n(x_0)\}$ converges to a fixed point of f if $x \in (0, \bar{x}_{\max}]$ and diverges to infinity if $x_0 \in (\bar{x}_{\max}, \infty)$.

Case 4.4.1.2.2. Assume that $B\gamma - C\beta < 0$.

As we observed earlier by examining equation (28), in this case f has a unique relative minimum at say μ. We will now investigate to see whether there exists a value $m < \mu$ such that $f(m) = \bar{x}_{\max}$, or equivalently

$$Am^2 + (B - \beta\bar{x}_{\max})m + C - \gamma\bar{x}_{\max} = 0. \tag{29}$$

Since we know $\bar{x}_{\max}$ is a solution of this equation, then $m - \bar{x}_{\max}$ is a linear factor. It is easy to check that the other linear factor is $Am + (A - \beta)\bar{x}_{\max} + B$. Thus, there is no non-negative solution to equation (29) for m. By Theorem 3.3, the solution $\{f^n(x_0)\}$ converges either to a non-negative fixed point of f or to a positive prime period-two pair of f if $x_0 \in (0, \bar{x}_{\max}]$, and diverges to infinity if $x_0 \in (\bar{x}_{\max}, \infty)$.

Observe that the results of Cases 4.4.1.2.1 and 4.4.1.2.2 give exception (d) in the conclusion of Theorem 4.

Case 4.4.2. Assume that $A = \beta$ and either $B > \gamma$, or both $B = \gamma$ and $C > 0$.

In this case, equation (27) has no positive roots and so there are no positive fixed points. Also, it is easy to see that $f(x) > x$ as $x \to \infty$. This implies that $f(x) > x$ for all $x > 0$. Obviously, for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ diverges to infinity. Observe that this case is exception (e) in the conclusion of Theorem 4.

Case 4.4.3. Assume that $A = \beta$, $B = \gamma$ and $C = 0$.

This is the trivial case $f(x) = x$.

Case 4.4.4. Assume that either $0 < A < \beta$ or both $A = \beta$ and $B < \gamma$.

It is easy to check that $f(x) < x$ as $x \to \infty$. From equation (27), it is obvious that there must be either one or two non-negative fixed points of f. By either Theorem 2.9 or Theorem 3.3, for every $x_0 > 0$ the solution $\{f^n(x_0)\}$ converges to a fixed point of f. □

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