COMPARING THE FAIRNESS OF TWO POPULAR SOLUTION CONCEPTS OF COALITION GAMES: SHAPLEY VALUE AND NUCLEOLUS[1]

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ABSTRACT

An analytics model is developed to compare two popular solution concepts of coalition games, and to determine which one is fairer and more appropriate to use in real life applications of coalition games. Coalition games have found wide application in economics, finance, politics, and computing. When applications are developed using coalition game theory, designers have the option of choosing from several solutions concepts and the effectiveness of the solution depends on the accuracy of the underlying solution concepts. The algorithm presented sheds light on the effectiveness and fairness of the solution concepts and, in particular, is used to compare the fairness between Shapley Value and Nucleolus solution concepts. When imputations for coalitions are expressed in a barycentric triangle, they create a core, and larger core areas leave room for more competition among the coalitions which can create instability. Smaller core areas are desirable because they leave little room for further collaboration and come to a quick resolution. An example supports the idea of smaller core comparison, and imputations have been calculated using Shapley Value and Nucleolus expressed in a barycentric triangle for comparison.

INTRODUCTION

Game theory has its foundations in applied mathematics. The mathematical models and built-in consistency of game theory make it a suitable framework and basis for modeling and designing of automated bargaining and decision-making software systems in interactive negotiation. For example, a game theory framework can serve as the most efficient bidding rules for Web Services, e-commerce auction website, or tamper-proof automated negotiations for purchasing communication bandwidth. Research in these applications of game theory is the topic of many recent conferences, journals, and scientific papers. The application of game theory to automated negotiation is still in a nascent stage. The automation of strategic choices enhances the need for these choices to be made efficiently.

Game theory is the formal study of conflict and cooperation [2]. These are decision problems with multiple decision makers, whose decisions impact one another and the outcome. Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact [3]. The basic assumptions that underlie the theory are that decision-makers pursue well-defined exogenous objectives (they are rational) and take into account their knowledge or expectations of other decision-makers' behavior (they reason strategically). An important characteristic of games is that the actions of one person have influence on the outcomes of other people in the game and vice versa.

Coalition Games have found application in economics, finance, politics, computing, and problem resolution of various kinds. Coalition game theory studies and models the joint activities of a group of decision makers who cooperate and take joint actions in a coalition to increase their stakes. The problem of distributing the jointly-earned wealth in a fair way as payoffs or, cost sharing among the coalitions,

[1] This paper is a summary of a doctoral dissertation by the first author as it nears completion.

remaining as a challenge till now. The game theoretic people have deduced a good quantity of methods of distribution, or solution concepts; and only a few of these solution concepts have been well accepted in the real life applications.

In this paper we discuss the coalition model extensively. Game theory requires that the conflict is modelled through proper analysis to find a proper solution to any conflict. We will highlight the differences between decision making of a single agent and interactive decision making, namely, a game. One of the differences is that in the first case we are aiming at getting an optimal decision. In games optimality does not always apply, and the result of games can be dismal even when all parties behave rationally.

This study investigates two important Solution Concepts of coalition games, and compares them to understand their accuracies in their calculation of imputations and in their fairness in application. The coalition games have several of these Solution Concepts, and among them Shapley Value and Nucleolus have found wide applications in computing, finance, economics, biology etc. Efforts have been made to understand the accuracy of these Solution Concepts in allocating of imputations gained by the Agents, participating in coalitions.

Solution Concepts of coalition games become complex as more than two players join a coalition, and each agent try to maximize their gains; so, when multiple coalitions try to compete and win, proper distribution of gains is important to hold the coalition in place. When software applications are developed using coalition game theory, the fairness of imputation is the key for accurate outcomes.

This study generalized the coalition game for more than two players. It also used methods to compare and examine the properties of Shapley Value and Nucleolus Solution Concepts to calculate imputations to see which values form the smaller core in barycentric coordinates, because when the area of the core is small the Solution Concept is more accurate and fair.

THE PROBLEM

Game theory provides formal tools and mathematical models to study strategic interactions among players. In strategic interaction situations, the assumption of common knowledge known as solution concepts holds. To study these situations and corresponding solution concepts we must resort to computational methods. Software is now available for coalition games, handling its underlying mathematical models and solution concepts. When applications are developed, designers have the option of choosing models and solutions concepts, and the effectiveness of the software depends on the accuracy of the underlying solution concepts.

In coalition game theory the modeler often must choose one of several substantively different solution methods, or solution concepts, which can lead to different game outcomes. Often one is chosen because it leads to a unique prediction [14]. In cooperative game theory, by defining solution concepts, one tries to characterize the set of outcomes that are, seen from a viewpoint of fairness and rationality. In this study we will describe and discuss the main solution concepts that have, in the course of time, been proposed by different game theorists. In the discussion we will also establish the relationship between two different concepts and their usefulness and limitations in real life applications.

An example will explain the necessity for using a fair solution concept. The New York City (NYC) public school system had a problem matching incoming freshmen to high schools where the school district students used to mail in a list of their five preferred schools in rank order to the Board of Education which then mailed a photocopy of that list to each of the five schools. The schools could then tell whether or not students had listed them as their first choice, but many schools only wanted first-choice students and this meant that some students really had a choice of only one school rather than five. To resolve this matching problem the school district went to experts and subsequently accepted the suggested Shapley's solution concept as the method of selection for NYC public school students [6]. But what would have happened if the experts selected the Nucleolus solution concept instead? Which solution concept would produce a fairer matching system? To answer such questions this study compares the level of fairness between these two solution concepts. In general, coalitions are outcomes of cooperation between players, are goal-oriented and short-lived, and are formed with a purpose in mind and dissolved when that purpose no longer exists or when players stop cooperation amongst themselves.

LITERATURE REVIEW

A game "is a competitive activity in which players contend with each other according to a set of rules" [10]. A game theory model can be designed to model any kind of conflict which is an interactive situation among the parties involved. If a conflict situation exists, there are multiple ways to resolve that conflict, and only few of them will be efficient. The application of game theory guarantees efficient resolution of the conflicts always, and also gives accurate mathematical reasoning of the situation. The nature and course of the conflict as well as its resolution depends on the decisions made by the various parties involved. Each party involved in the conflict, when considering its decisions, should take into account the decisions made by all the other parties. Game theory is the formal study of conflict and cooperation [12]. Games are decision problems with multiple decision makers, whose decisions impact one another. Game theory is a bag of analytical tools designed to help us understand the phenomena that we observe when decision-makers interact [11].

Professor Bruce Bueno de Mesquita of New York University recently made a handful of very impressive and accurate political predictions: in May 2010, he predicted that Pervez Musharraf, the Pakistani president had to leave his office by the end of summer of that year. Pervez Musharraf was forced out of his office before the end of the year. Bruce also successfully predicted the fall of Egyptian president, Hosni Mubarak. Approximately five years before the death of Ayatollah Khomeini in 1989, Bruce accurately predicted the name of his successor, and, since then, Bruce has made many dozens of consistent forecasts as a consultant to State Department, Pentagon and intelligence agencies and few foreign governments. The secret behind his success is the application of game theory and related mathematical models.

In 1962, Shapley together with David Gale wrote two papers on two important classes of games: cooperative and coalition games; the paper with title "College Admissions and the Stability of Marriage," authors studied matching games with nontransferable utility. The canonical example is the marriage game, where m women are to be matched with m men. A matching is "stable" if there is no woman and man who prefer each other to their current match. Gale and Shapley showed that the set of stable matching is precisely the set of core allocations, proved that there exists a stable matching, and provided an algorithm for finding it. In another 1962 paper, Martin Shubik and Shapley studied a transferable utility version of the matching game (called an assignment game). These two games have become workhorses in the study of labor markets and other two-sided markets and other two-sided markets.

In his 1953 paper, "A Value for N-Person Games," Shapley provided a set of axioms that, in every cooperative game, uniquely identified a payoff for each agent. This payoff has come to be called the Shapley value. The paper has been enormously influential, both through the widespread use of the Shapley value, and by inspiring other values obtained through modifications to Shapley's original axioms. The Shapley value has been used to quantify the impact of voting rules on the influence of individual voters, where it is often called the Shapley-Shubik index (after a 1954 Shapley and Shubik paper that analyzed the influence of different members of the United Nations Security Council). This measure is of more than academic interest: A related measure (the Banzhof index) has played a role in legal decisions concerning electoral districting and representation. The Shapley value has also been widely applied in accounting to problems of cost allocation.

Von Neumann and Morgenstern who were the originators of multi-person cooperative games proposed their own solution concept, known as the Stable Set [13]. In 1953, Gillies introduced the concept of core as the set of all un-dominated payoffs (i.e., imputations) to the players satisfying rationality properties. Even though the core has been found useful in studying economic markets, it does not provide a unique solution to the allocation problem. Also in 1953, Shapley wrote four axioms which would capture the idea of a fair allocation of payoffs and developed a simple, analytic, expression to calculate the payoffs. Shapley value can be computed easily by using a formula regardless of whether or not the core is empty [4]. However, when the core is non-empty, Shapley value may not be in the core and under some conditions the allocation scheme in terms of Shapley value may result in an unstable grand coalition. An alternative solution concept known as the nucleolus was introduced by another game theoretic, Schmeidler. In 1969, Schmeidler defined and introduced the understanding of Nucleolus. The nucleolus of a coalition game is the imputation with lexicographically minimal excess, based on ordering of the imputations. The nucleolus solution is defined as an n-tuple of imputation x = (x1, x2,…xn) such that the excess of any possible coalition S cannot be lowered without increasing any other greater excess [1].

Solution concept is the term used to mean the methods used to divide the cost or payoffs fairly amongst the participating players who acted jointly in the coalition. In the process one should choose an allocation scheme that satisfies coalition’s joint agreement with natural monotonicity property of the game. Monotonicity of a scheme is defined as when the cost or payoffs incurred by each possible coalition rises, then the cost or payoffs allocation to each entity under the scheme should also be increased accordingly. The nucleolus is not always monotonic [1].

Game theory became a popular area of research due to its mathematical models and wide application in the fields of computing, economics, finance, politics, stock markets, prediction markets, international relationships, and generally any field where multiple parties were involved in making decisions. Application software is available using the underlying principles of coalition game and its underlying mathematical models using solution concepts. Many contributions were made by academicians in this area, some of which are very complicated in nature.

The solution concepts, which are the logical rules of sharing the imputations, play an important role to hold the coalitions and share their worth. Two of the six main solution concepts, namely Shapley Value and Nucleolus, have been widely used by prominent game theoreticians because of their simplicity and accuracy.

In coalition games, coalitions stay together when their payoffs are maximum and also fair compared to other possible coalitions. The coalition members act together to maximize their shares, or imputations, and if the imputations are not sufficiently fair, i.e. if members are dissatisfied, the coalition will not exist and fairer methods should be applied in the allocation process. Here, we determine which solution concept is best by comparing their outputs. For comparing, we have used: (1) the deviations, in which the outputs have been calculated with pro-rata outputs, then compared to see which one produce minimum deviations; (2) more accurately, comparisons by finding the core created by Shapley Value and Nucleolus in barycentric triangles to find a solution concept producing the smaller Core; (3) also comparisons by using lexicographic ordering.

COALITION GAMES AND SOLUTION CONCEPTS

A coalition game is a special case of cooperative games where individual players cooperate among themselves as part of a coalition on some joint activities and decision making. Coalitions are formed for mutual benefit with each agent or player of a coalition trying to maximize their payoffs and the payoffs of the coalition. The central understanding of a coalition game is the formation of the coalitions and the rules governing the existence and activities of coalitions. Coalitions are partitions on the super set of total players in the game. The binding force of a coalition is the worth of the coalition. The solution concept will predict the coalition formations and the final allocation (payoffs) to participants. A coalition game consists of two elements: (i) a group of players who participate in the game and (ii) a characteristic function specifying the value created by different subsets of the players in the game. Formally, let N = {1, 2 … n} be the sets of players and n=|N| the total number of players. The characteristic function is a function, denoted v(S), that associates with every subset S of the n players the value created when the members of S work together. In summary, a coalition game is a pair (N, v), where N is a set and v(S) is a characteristic function [2]. Solution concepts like Shapley Value are used to calculate the worth at the individual marginal level using the characteristic function v(S) of the game (N,v). A set consisting of all the players in a coalition game is known as the Grand Coalition, and the characteristic function for the grand coalition is v(N). The players form coalitions in any form and the solution concept will predict the coalitions that form and the final allocation (payoffs) to all coalitions and ultimately to the players.

The basic idea behind coalition game theory is that the players will be benefit by forming coalitions, and each member's payoff will be better or equal to the payoff when the player is acting alone. In a cooperative game model, the formation of the grand coalition, N, will lead to the highest benefit of the coalition since by super additivity the amount received, v(N), is always better, or equal to the total amount earned by any number of disjoint set of coalitions they could form. We will prove that it is reasonable to suppose that rational players will always agree to form the grand coalition and receive maximum payoff v(N). A grand coalition always guarantees the best payoffs, and follows rules to split the amount reasonably among the players. We will discuss one of the possible properties of an agreement on a fair division, that it be stable in the sense that no coalition should have the desire and capability to upset the agreement of forming the coalition. This understanding is known as 'core' in coalition game theory, and is used widely in all forms coalition game theory. To understand core, we need to introduce another term called the imputation.

Imputations are distributions of payoff vectors at the individual marginal level which are efficient and individually rational. Imputation can be expressed with the help of payoff vectors. For a coalition game, an imputation or payoff vector $\boldsymbol{x}$ = ($x1$, $x2$, . . . x_n) for a coalition S is a proposed amount distributed among the players, such that player i is to receive x_i. The total amount received by the players in the coalition is v(S), where S is a subset of N. The set of imputations may be written as: {**x** = (x1, . . , xn) : i $\sum_{i \in N} xi$ = v(N), and xi ≥ v({i}) ∀ i ∈ N}. To satisfy the imputation conditions, a payoff vector has to be group rational. If the payoff vector is expressed as x = (x1, x2, . . . , xn), the x will be group rational or efficient if and only if $\sum_{i=1}^{n} xi$ = v(N). A player participating in a coalition will always expect to receive a better payoff than could be earned acting alone. This essential condition for imputation can be expressed as follows: **x** = (x1, x2, . . . , xn), is that xi ≥ v({i}) for all players i in the coalition [2].

The Shapley value is one of the popular solution concepts for dividing the worth of the coalition. The Shapley value is characterized by a collection of desirable properties known as Shapley Axioms. The Shapley Value imputation for the i^{th} player is given by [5]:

$$\emptyset i(v) = \sum_{S \in N\{i\}} \frac{(|S|-1)!\,(n-|S|)!}{n!}\left(v(S \cup \{i\}) - v(S)\right)$$

The Nucleolus solution concept is defined as an n-tuple imputation x = (x1, x2,…xn) such that the excess (unhappiness) of any possible coalition S cannot be lowered without increasing any other greater excess until all coalition reaches to a satisfaction level. The nucleolus of a coalition game is the imputation with lexicographically minimal excess, based on ordering of the imputations [9].

Some additional solution concepts will be described briefly. In a Core Concept solution the core of the coalition game is the set of actions A of the Grand Coalition N such that no coalition has an action that all its members prefer to action A[3]. In coalition game, the core is defined as the set of all feasible allocations that cannot be improved upon by a coalition. An imputation is said to be in the core if the imputation is equal or better than the individual pay-offs. A coalition is said to improve upon or block a feasible allocation if the members of that coalition are better off under another feasible allocation. In a Kernel solution [8] concept payoff, all players are in a sort of "bilateral equilibrium", in the sense that the threats to each other are equalized [10]. The definition seems to involve interpersonal utility comparisons. The Stable Set solution concept was suggested by Von Neumann and Morgenstern in 1944, in their book, Theory of games and economic behavior. The stable set solution concept [8], also known as the von Neumann-Morgenstern solution dealt with the games with more than 2 players. A stable set is a set of imputations that satisfies two properties: (a) Internal stability (b) No payoff vector in the stable set is dominated by another vector in the set, means no other imputations can block the coalition [14]. External stability: All payoff vectors outside the set are dominated by at least one vector in the set. Von Neumann and Morgenstern saw the stable set as the collection of acceptable behaviors in the society. The Banzhaf power index is a power index defined by the probability of changing an outcome of a vote where voting rights are not necessarily equally divided among the voters or shareholders. This solution concept was invented by Lionel Penrose in 1946, was named after John F. Banzhaf III and is sometimes called the Penrose–Banzhaf index [14].

SHAPLEY VALUE SOLUTION CONCEPT

This solution concept method calculates a single payoff for each player in the coalition, the average of all marginal contributions of a player in the coalition. The basic idea is that those who contribute more to the coalitions that include them should earn more.

A value function, φ assigns to each possible characteristic function of an n-person game, v, a vector of real numbers $\emptyset i(v) = (\emptyset 1(v), \emptyset 2(v), \dots \emptyset n(v))$. Here $\emptyset i(v)$ represents the worth or value of the ith player in the game with characteristic function $v(S)$.

The Shapley Axioms for $\varphi(v)$[2] :

1. **Efficiency:** This is also known as Group Rational, or Collectively Rational; **t**he total worth distributed among the players: $\sum_{i \in N} \emptyset i(v) = v(N)$.
2. **Symmetry.** If i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for every sub-coalition S in N, and S not containing i and j, then $\emptyset i(v) = \emptyset j(v)$. Two participants, i, j are said to be symmetric with respect to coalition game (N, v) if they make the same marginal contribution to the coalition.
3. **Dummy Axiom.** If i is such that $v(S) = v(S \cup \{i\})$ for every coalition S not containing i, then $\emptyset i(v) = 0$
4. **Additivity.** If u and v are characteristic functions, then $\varphi(u + v) = \varphi(u) + \varphi(v)$.

The first axiom syas that the total value earned the players is the total value of the grand coalition. For a coalition to act together, this is the basic rule, as each participant will do better in the coalition than individually. The second axiom says that if the characteristic function is symmetric for two players i and j, then the values assigned to i and j should be equal, i.e. v({i}) = v({j}). The third axiom states that the contribution by a dummy participant is null and his payoff vector is null. The forth states that when two games are played at the same time the sum of the values of the games are as if they are played separately.

The Shapley value is used to distribute the total gains to the players, provided that the players were the part of the coalition. For a game (N, v) the distribution is said to be fair in the sense that it satisfies the following condition, $\varphi_i(v) \geq v(\{i\})$, that the players' payoff will be a fair portion of the payoffs of the Grand Coalition.

If the total number of participants in the game is n, then a coalitional is expressed by the pair (N, v), where N = {1, 2,, . . . , n} is the set of players and v is called the characteristic function of the game, which determines the worth, defined on the set, 2^N, of all possible combinations of coalitions, i.e. the subsets of N. The formula is used to calculate the average marginal contribution by individual player, i to each possible coalition, (S1, S2, …Sn) those can be formed from the players. To form the coalition, one player enters at a time; so, there exists N! ways to do this. Now, during the formation of coalition S, we can do it in S! ways. The characteristic function will be v(S). As player i-th enters the coalition S, the new value of the characteristic function will become v(S U {i}).

Player i is a member of the coalition S; so the quantity, v(S) − v(S − {i}), is the amount by which the value of coalition (S − {i}) increases when player i joins the coalition. Now, to find value φi(v), we need to list all coalitions containing i, sum up all the value of player i's contribution to that coalition, then multiply by value ((|S| − 1)!(n − |S|)!/n!), and ultimately take the sum. We have to choose a random order of the players with all n! permutation orders of the players equally likely. Next, we have to determine positional worth of the player as follows: if, when player i enters the coalition, he forms coalition S and he finds coalition (S − {i}) is already there, he receives the amount [v(S) − v(S − {i})]. Now, we have to compute the probability of finding the coalition S − {i} in front of him; and it should be (|S| − 1)!(n − |S|)!/n!. The denominator of the expression is the total number of possible permutations of the n participants. The numerator is number of these permutations in which the (|S| −1) members of (S −{i}) coalition come first in ((|S| − 1)!) ways; then player I; and then the remaining (n − |S|) players in ((n − |S|)!) ways. So, this formula shows that the amount φi(v) is just the average amount player i contributes to the grand coalition if the players sequentially form this coalition in a random order [2]. The Shapley Value for player i-th player is given by:

$$\emptyset i(v) = \sum_{S \in N\{i\}} \frac{(|S|-1)!\,(n-|S|)!}{n!}\left(v(S \cup \{i\}) - v(S)\right)$$

NUCLEOLUS SOLUTION CONCEPT

The nucleolus of a cooperative game is a solution concept that makes the largest unhappiness of the coalitions as small as possible, or, equivalently, minimizes the worst inequity. In 1969, Schmeidler came up with a new solution concept known as the nucleolus; he proposed an allocation scheme that minimizes the "unhappiness" of the most-unhappy coalition during distribution of payoffs. Schmeidler defines "unhappiness" or excess of a coalition as the difference between what the members of the coalition could get by themselves and what they are actually getting if they accept the allocations suggested by a solution. It was shown by Schmeidler that if the core for a cooperative game is non-empty, then the nucleolus is always located inside the core and thus assures stability of the grand coalition [7]. For calculating Nucleolus, there does not exist any established formula; in most of the cases, it is computed analytically, or numerically in an iterative manner by solving linear programming (LP) problems.

In cooperative games, the players jointly coordinate their actions by forming multi-level coalitions. The most important steps in the process is the fair distribution of the worth among all coalition members generated from such a joint effort. The Shapley Value and Nucleolus are widely recognized and used as a fair way of distributing the gain in a coalition. There exist efficient algorithms for computing the nucleolus using linear programming based techniques; the implementation of those methodologies are infeasible in multi-player coalitions as the burden of decision making is distributed among all players in the coalition, and there is no single decision market that can aggregate all data and compute for all.

As a solution concept, the nucleolus is very important and well accepted solution in cooperative game theory despite calculation difficulties. The nucleolus satisfies some important desirable properties, e.g. if core is not empty, it exists uniquely in the core; and is considered as an important fair division scheme [7]. Many researchers have used this concept to analyze computational, business and management problems, though the complexity of calculations made the nucleolus a tough choice in many resource allocation applications.

For fairly distribution of cost or payoffs among multiple players, one should choose an allocation scheme that satisfies a natural monotonicity property of the coalition. In the context of cost sharing, the monotonicity of an allocation scheme means that, if the cost or worth incurred by each possible coalition rises, then the cost or worth allocation to each coalition under the scheme should be increased. The nucleolus is not always monotonic which is considered as a drawback of this concept [7].

In many cost allocation problems, when the cost for using the common resource increases, then the nucleolus solution may suggest a lower cost allocated to some entities, which means that the nucleolus is not monotonic always. Other solution concepts those satisfy the monotonicity property can be used instead of the nucleolus, or they can be used in combination. Grotte introduced a new method of calculating the Nucleolus; he normalized the nucleolus by dividing the excess of each coalition by the number of players in the coalition; this new concept, he name per capita normalized nucleolus. Grotte proved that the per capita nucleolus is monotonic and also always exists in the core [7].

When nucleolus is calculated, it can be readjusted to be in the core, if the core is not empty. The nucleolus satisfies the symmetry axiom and the dummy axiom; it is group rational and individually rational. The only difficult part to prove is the uniqueness of the nucleolus. Since the nucleolus always exists and is unique, we may speak of the nucleolus of a game. Like the Shapley value, the nucleolus will satisfy individual rationality if the characteristic function is super-additive or, more generally, if it is monotone in the sense that for all players i and for all coalitions S not containing i, we have $v(S)+v(\{i\}) \leq v(S \cup \{i\})$. In contrast to the Shapley value, the nucleolus will be in the core provided the core is not empty. The nucleolus satisfies the first three axioms of the Shapley value [2].

Excess, an inequity measure of an imputation $\boldsymbol{x}$ for a coalition S, is defined as: $e(\boldsymbol{x}, S) = v(S) - \sum_{J \in S} \boldsymbol{x_j}$. Excess measures the amount or the size of the inequity by which coalition S falls short of its potential v(S) in the allocation x. Since the core is defined as the set of imputations such that for all coalitions S, we immediately have that an imputation x is in the core if and only if all its excesses are negative or zero. During the distribution of worth, the coalition which complains that it is not getting its proper share, efforts will be made to give it a fair share.

For calculating Nucleolus, there does not exist any established formula; in most of the cases, it is computed analytically, or numerically in an iterative manner by solving linear programming problems. The steps of finding the nucleolus is to find a vector $\boldsymbol{x} = (x1, x2, \ldots, xn)$ that minimizes the maximum of the excesses $e(\boldsymbol{x}, S)$ over all S subject to $xj = v(N)$. This problem of minimizing the maximum of a collection of linear functions subject to a linear constraint is easily converted to a linear programming problem. After this is done, one may have to solve a second linear programming problem to minimize the next largest excess, and so on [7]. We will follow the analytical method.

From the definition of Nucleolus, it is clear that the nucleolus of a cooperative game is a solution concept that makes the largest unhappiness of the coalitions as small as possible, or, equivalently, minimizes the worst inequity. Using analytical method, which is based on lexicographic ordering we reduce the largest excess of the imputations, for all coalitions as much as possible, then decrease the second largest excess as much as possible, and continue this process until the n-tuple imputation x is determined [7].

The nucleolus is computed by applying linear programming in multiple steps:

Step 1: In this step, possible coalitions are formed, empty coalition is omitted.
Step 2: Using the characteristic functions, pro-rata imputations are calculated.
Step 2: In this step, individual coalition's joint worth is assigned to the coalitions.
Step 3: At this stage, the excess e(x,S) is calculated.
Step 4: Now using the table, one needs to find next vector $\boldsymbol{x} = (x1, \ldots, xn)$ that minimizes the maximum of the excesses $e(\boldsymbol{x}, S)$ over all coalitions S subject to $v(N)$.
Step 5: Now using the table, one needs to find next vector $\boldsymbol{x} = (x1, \ldots, xn)$ that minimizes the maximum of the excesses $e(\boldsymbol{x}, S)$ over all coalitions S subject to $v(N)$.
Step 6: Step 4 is repeated until all imputations are processed; and the excess are minimized.

MEASURING FAIRNESS

We will now focus on a game with three players and look into the imputations generated by Shapley Values and Nucleolus.

Assumptions and representations:
(1). We will represent the coalition as: {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}
(2). Characteristic functions: v(1), v(2), v(3), v(1, 2), v(1,3), v(2, 3), v(1,2,3) : this characteristic functions expresses a 3 player coalition game.
(3). Each player's marginal share in the coalition is called the imputation; players imputations will be represented by: x1, x2, x3 satisfying: x1>= v(1), x2 >= v(2) and x3 >= v(3). Also, x1 + x2 + x3 = v(1, 2, 3) [2].
(4). When all imputations are in the core, they are considered as 'fair' and none of the coalition will try to block the imputations in the core. The imputations will be in the core, iff

- (1) x1 + x2 >= v(1,2)
- (2) x1 + x3 >= v(1,3)
- (3) x2 + x3 >= (v(2,3)

(5). Symmetrical players: Two players i & j are symmetrical when characteristic function v(ik) = v(jk) with assumption that k's marginal imputation is the same on the both coalition.

There are various methods of measuring fairness. Shapley value and Nucleolus are popular Solution Concepts for coalition game based solutions, applications and models. Coalition games based applications and predictions models needs effective models with fair Solution Concepts. Our algorithm will help to understand the effectives and fairness of the imputations used deducted using Shapley Value and Nucleolus.

1. Comparing Deviations: We will use deviations of imputations from pro-rata, Shapley value, and Nucleolus; imputations having larger deviations are less fair.
2. Lexographic ordering : We will use lexographic order for comparing Shapley Values with Nucleolus values.
3. Smaller Core: We will use Core capabilities of the imputations, i.e. which values create a compact smaller Core.

For comparing deviations we have the steps:
(1). Take the given coalition game for 3 players (|N| =3), write it in terms of characteristic functions, v(1), v(2), v(3), v(12), v(1,3), v(2,3), v(1,2,3); and imputations: x1, x2, x3.

(2) Calculate pro-rata imputations for individual players and coalitions.

$$x_1^{PR}, \quad x_2^{PR}, \quad x_3^{PR}$$

(3) Calculate imputations using Shapley Value Solution concepts:

x_1^S, x_2^S, x_3^S

(4) Calculate imputations using Nucleolus Solution concepts:

x_1^N, x_2^N, x_3^N

(5) Next, calculate the deviations of imputations between the pro-rata imputations and Shapley Value imputations; and then, between the pro-rata imputations and Nucleolus imputations. Now, compare the deviations, if are positive, and do not exceed Grand coalitions payoffs, then the bigger imputations are more fair than the other.

$x_1^{DS} = x_1^{PR} - x_1^S$
$x_2^{DS} = x_2^{PR} - x_2^S$
$x_3^{DS} = x_3^{PR} - x_3^S$
$x_1^{DN} = x_1^{PR} - x_1^{\mathrm{N}}$
$x_2^{DN} = x_2^{PR} - x_2^{\mathrm{N}}$
$x_3^{DN} = x_3^{PR} - x_3^{\mathrm{N}}$

For the lexicographical ordering method we use the lexographical order. We say, for any vector y =(y_1,..., y_k) is lexographically less than a vector z = (z_1,...,z_k), and write y <L z, if $y_1 < z_1$, or if $y_1 = z_1$, or and $y_2 < z_2$, or if $y_1 = z_1$, $y_2 = z_2$ and $y_3 < z_3$, or..., or if $y_1 = z_1, \ldots,$ $y_{k-1} = z_{k-1}$ and $y_k < z_k$ [2]. That is, y <L z if in the first component in which y and z differ, that component of y is less than the corresponding component of z. Similarly, we write y ≤L z if either y <L z or y = z.

For the smaller core method we use the core concept together with barycentric coordinates to compare the imputations: if core imputations create triangles with smaller area, the distribution will be more accurate and less prone to fluctuate; a bigger area of the triangle indicates less fairness, which is the cause of dissatisfactions.

EXAMPLE: COALITION OF THREE NEIGHBORING FARMS

This is a three-player coalition game involving three neighboring farms connected to each other and to the main highway by a series of trails. The farms are planning to build paved roads connecting them to the highway. They can build them individually, or by forming coalitions as shown in Figure 1.

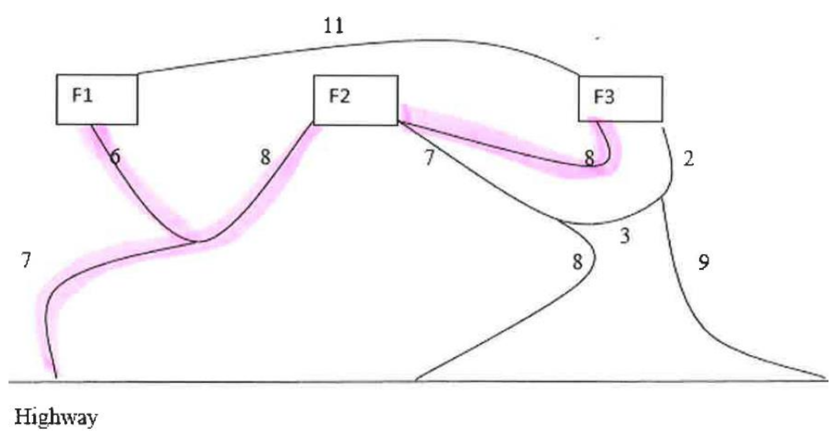

Figure 1. Logical connections and estimated paving costs in millions of dollars for calculating coalition benefits for farms F1, F2, and F3. The optimal road paving solution is highlighted in red.

Possible Coalitions:
{∅}, {F1}, {F2}, {F3}, {F1, F2}, {F1, F3}, {F2, F3}, {F1, F2, F3}

Opportunity Cost:

We define the opportunity cost for each player and all possible coalitions. An opportunity cost is defined as the minimum cost of a player or coalition. The opportunity cost in this example are: oc(F1) = 13, oc(F2) = 15, oc(F3) = 11, oc(F1,F2) = 21, oc(F1,F3) = 22, oc(F2,F3) = 20, oc(F1,F2,F3) = 29.

Possible coalitions:

(1) Players F1 and F2 can build the desired road for v(F1, F2) = 21 million dollars. If built separately, the farms have to spend 28 million. Jointly the farms save = 7 million.
(2) Players F1 and F3 can build it together for v(F1, F3) = 22. If built separately, the farms have to spend 24 million. Jointly the farms save 2 million.
(3) Players F2 and F3 can build it for v({F2, F3}) = 20. If built separately, the farms have to spend 26 million. Jointly the farms save 6 million.
(4) All 3 players: F1, F2, F3 i.e. the grand coalition can build for v({F1,F2, F3}) = 29. If build separately, the farms have to spend 39 million. Jointly the farms save 10 million.

Characteristic functions:
V(F1) = 0, V(F2) = 0, V(F3) = 0, v({F1, F2}) = 7, v({F1, F3}) = 2, v({F2, F3}) = 6, v({F1,F2, F3}) = 10

We now find the pro-rata profits for the players, where the players share the savings proportional to their opportunity costs. The pro-rata distribution is a simple distribution and not considered a solution concept but rather assumes that the farms form a hypothetical Grand Coalition which gives proportional incentives to all players. The pro-rata distribution to the farms is:

(1) F1 distribution: [29 * 13/ (13 + 15 + 11)] = 9.67 and savings = 13 – 9.67 = 3.33
(2) F2 distribution: [29 * 15/ (13 + 15 + 11)] = 11.15 and savings = 15 – 11.15 = 3.85
(3) F3 distribution: [29 * 11/ (13+ 15 + 11)] = 8.17 and savings = 11 – 8.18 = 2.82

Pro rata imputations: (3.3, 3.9, 2.8)

Computing Shapley Value

The coalitions and characteristic function are as above.

We find inductively:

c{F1} = v({F1}) = 0, c{F2} = v({F2}) = 0, c{F3} = v({F3}) = 0, c{F1,F2} = v({F1,F2})−c{F1}−c{F2} = 7 – 0 − 0 = 7
c{F1,F3} = v({F1,F3}) − c{F1} − c{F3} = 2-0-0 = 2 and c{F2,F3} = v({ F2,F3})-c{F2}-c{F3} = 6-0-0 =6
Finally,
C{N} = v({N}) - c{F1,F2} - c{F1,F3} - c{F2,F3} - c{F1} - c{F2} - c{F3} = 10 - 7 - 2 - 6 - 0 - 0 - 0 = -5
Hence, we know we can write v as
v = w{1} + w{2} + w{3} +7w{1,2} + 2w{1,3} + 6 w{2,3} -5w{1,2,3} or
v = 7w{1,2 } + 2w{1,3}) + 6w{2,3} -5w{1,2,3} (omitting 0 values)
Now, we find φ1(v), φ2(v), φ3(v):

$\varphi 1(v) = \frac{7}{2} + \frac{2}{2} - \frac{5}{3} = \frac{17}{6}$

$\varphi 2(v) = \frac{7}{2} + \frac{6}{2} - \frac{5}{3} = \frac{29}{6}$

$\varphi 3(v) = \frac{2}{2} + \frac{6}{2} - \frac{5}{3} = \frac{14}{6}$

Shapley Value imputations: (17/6, 29/6, 14/6) = (2.8, 4.8, 2.3)

Computing Nucleolus

The coalitions and characteristic function are as above.

Table 1. Tabular system for calculating Nucleolus

Coalition	v(S)	e(x,S)	(3.3, 3.9, 2.8)	(3.5, 3.9, 2.6)	(3.5, 3.9, 2.6)
{F1}	0	-x1	-3.3	- 3.5	-3.5
{F2}	0	-x2	-3.9	-3.9	-3.9
{F3}	0	-x3	-2.8	-2.6	- 2.6
{F1,F2}	7	7 –x1- x2	- 0.2	- 0.45	-1.4
{F1,F3}	2	2 –x1 –x3	-4.1	-4.1	-4.1
{F2, F3}	6	6 –x2 –x3	- 0.7	- 0.45	-0.5

Step 1: To find imputations, we create a matrix with the pro-rata imputations; here, the imputations will be (3.3, 3.9, 2.8).
Steps 2: We determine the vector of excesses of imputations denoted by e = (−3.3, −3.9, −2.8, 0.2, −4.1, 0.7).
Step 3: We look for the first dissatisfied coalition by identifying largest of these values of the vector, and it is -0.2, for the coalition (F1, F2). This coalition will be dissatisfied as the other coalitions are doing better than it is. So, we have to improve the excess of this coalition by making it smaller. Also here, we will look for nearest large value, which will be used to normalize the imputations; that value is -0.7.
Step 4: The next large imputation is -0.7 for coalition (F2, F3); and it is the closer value. We will try to normalize the values of these two coalition such that excesses e(**x**, F2, F3) = e(**x**, F1,F2); to do that we will increase x1 and decrease x3; and they will meet at -0.45 By solving the excesses for the two coalitions, we find that x1 = 3.5 and x3 = 2.6.
Step 5: Next vector of excess is (−3.5, −3.9, −2.6, 0.45, −4.1, 0.45). At this stage, we have two stable values calculated from the last steps: x1 = 3.5 and x3 = 2.6. The third value x2 = 10 – x1 –x3 gives us 3.9, which is the current value
Step 6: Once, x1 and x3 have been calculated, we recalculate the new values for the coalition by using values of x1 & x3 . New imputation vector is (−3.5, −3.9, −2.6, −1.4, −4.1, −0.5).
Step 7: Now, we will recalculate the value of x2; we will try first equation (6 –x2 –x3 = -0.45). Here, if we use the current value of x3 = 2.6, we have x2 = 3.9.
Step 8: By solving another equation (7 – x1 –x2 = - 0.45), we try to see whether there will be better value for x2; it gives x2 = 4.0; which actually may upset other two values: x1, x3.
Step 9: So, we will go with the imputations (3.5, 3.9, 2.6)

Nucleolus imputations: (3.5, 3.9, 2.6)

Comparing Fairness

1. **Deviations of imputations:**
   Pro-rata imputations of the coalitions: {(1,2), (1,3), (2,3)}: (43, 37, 40)
   Shapley value imputations of the coalitions: { (1,2), (1,3), (2,3) }: (46, 31, 43)
   Deviation for Shapley Values: **(-3, 5, -3)**
   Pro-rata imputations of the coalitions: { (1,2), (1,3), (2,3) }: (43, 37, 40)
   Nucleolus imputations of the coalitions: { (1,2), (1,3), (2,3) }: (43, 37, 40)
   Deviation for Nucleolus Values: **(0, 0, 0)**

2. **Lexicographical Ordering of imputations as coalitions:**
   Imputations of coalitions according to Shapley Values: {(1,2), (1,3), (2,3) } (46, 31, 43)

Imputations of coalitions according to Nucleolus values: {(1,2), (1,3), (2,3) } (43, 37, 40)
Lixicographical order, S <>l N, for
Set S{46, 31, 43} and set N{43, 37, 40} in mixed state.

3. Smaller Core:

(a) Finding the Core using Shapley Value imputations:

We have the imputations: (17/6, 29/6, 14/6)

The core created by Shapley Value imputations:

The triangle at the middle of the barycentric triangle, vertices: (0, 0, 60), (60, 0, 0), (0, 60, 0) created by coalition imputations represent the core.

The coalition imputations are represented by the barycentric equilateral triangle ABC and the triangle xyz at the middle of the barycentric triangle ABC (vertices: (0, 0, 60), (60, 0, 0), (0, 60, 0)) represents the core (Figure 2). The area of triangle xyz (the core) = area of triangle ABC – (area of trapezoid Amyq + area of trapezoid Bozm + area of trapezoid Coxq). Herons' formula will be used to find the area of the triangle. Heron's formula: area of triangle = sqrt(s(s -a) (s -b) (s -c)) ; semi-perimeter, s = (a +b +c) /2.

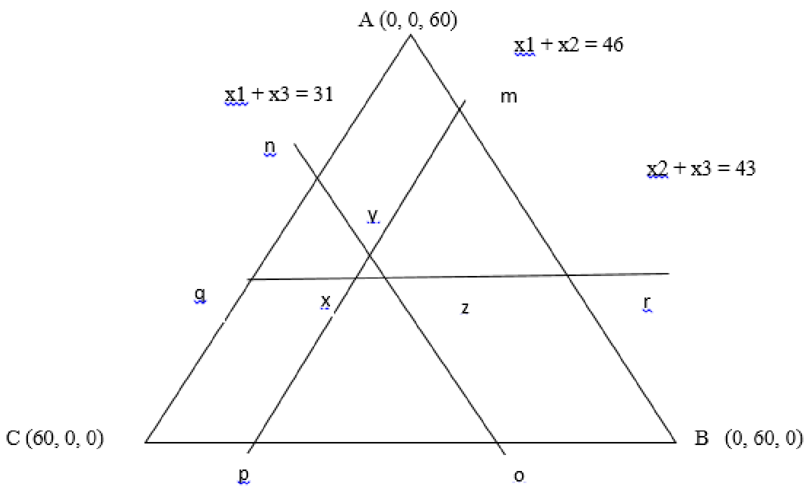

Figure 2. The Shapley Value barycentric triangle ABC and the core triangle xyz.

Line mp intersects sides AB at m and BC at p respectively; line mp also intersects line qr at the point y, and line on at the point z respectively. Line qr intersects sides AB at the point r and AC at q respectively; line qr also intersects line mp at the point y, and line on at the point x respectively. Line on intersects sides AC at the point n and BC at o respectively; line on also intersects line qr at the point x, and line mp at the point y respectively. Area triangle ABC = sqrt(s(s -a)(s -b)(s -c)), Herons' formula =sqrt(90 * 30 * 30 * 30) = 1558.845 sq units. Area of triangle xyz(the core) = area of triangle ABC – (area of trapezoid Amxq + area of trapezoid Boym + area of trapezoid Cqzo). Height of ABC triangle = 51.96 units.

Area of trapezoid Amxq = (((29 + 43)/2) * h) = 36 * 12.123 = 436.464 sq units
Area of trapezoid Boym = (((17 +46)/2) * h) = 31.5 * 25.114 = 791.091 sq units
Area of trapezoid Cqzo = (((14 +31)/2) * h) = 22.5 * 14.721 = 331.222 sq units
Area of core = 1558.845 – (436.428 + 791.091 + 331.222) sq units = 1558.845 - 1558.741 = 0.104 sq units

(b) Finding Core using Nucleolus imputations:

Nucleolus Values: (3.5, 3.9, 2.6)

The core created by Nucleolus Value imputations:

The triangle at the middle of the barycentric triangle with vertices: (0, 0, 60), (60, 0, 0), (0, 60, 0) created by the coalition imputations represents the core (Figure 3). Triangle ABC is equilateral with sides: a = 60, b = 60, c = 60.

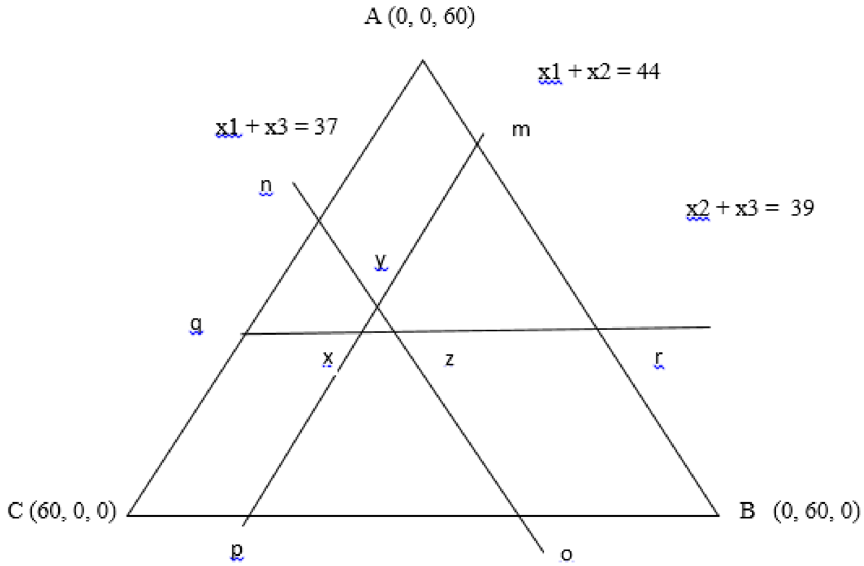

Figure 3. The Nucleolus Value barycentric triangle ABC and the core triangle xyz.

Line mp intersects sides AB at m and BC at p respectively; line mp also intersects line qr at the point y, and line on at the point z respectively. Line qr intersects sides AB at the point r and AC at q respectively; line qr also intersects line mp at the point y, and line on at the point x respectively. Line on intersects sides AC at the point n and BC at o respectively; line on also intersects line qr at the point x, and line mp at the point y respectively. Area triangle ABC = sqrt(s(s -a) (s -b) (s -c)) Herons' formula =sqrt(90 * 30 * 30 * 30) = 1558.845 sq units. Area of triangle xyz(the core) = area of triangle ABC –(area of trapezoid Amxq + area of trapezoid Boym + area of trapezoid Cqzo). Height of ABC triangle = 51.96 units, perimeter p = (a +b +c) /2. Area triangle ABC = sqrt(p(p-a) (p-b) (p-c)) Herons' formula = sqrt(90 * 30 * 30 * 30) = 1558.845 sq units.

Area of trapezoid Amxq = (((23+ 39)/2) * h) = 31 * 13.856 = 429.536 sq units
Area of trapezoid Boym = (((21 +44)/2) * h) = 32.5 * 19.919 = 647.367 sq units
Area of trapezoid Cqzo = (((16 +37)/2) * h) = 26.5 * 18.167 = 481.425 sq units
Area of core = 1558.845–(429.536+481.425+647.367) = 1558.845-1558.326 = 0.517 sq units

Comparing the areas of the cores:
Area of the Core for Shapley Values = 0.104 sq units
Area of the Core for Nucleolus = 0.517 sq units

Comparing fairness:
For this example, a summary of the imputations (payoffs) for the three methods is shown in Table 2.

Table 2. Summary of the payoffs to the farms in millions of dollars for the three methods.

	Farm 1	Farm 2	Farm 3
Pro Rata	3.3	3.9	2.8
Shapley Value	2.8	4.8	2.3
Nucleolus	3.5	3.9	2.6

The simple pro-rata payoff method, although not considered a solution concept, can serve as a baseline for comparison with the two solution concept methods. The Shapley Value solution concept method gives higher payoffs to the players contributing more to the coalition, so in this case Farm 2 was given a substantially higher payoff relative to the Pro Rata payoffs because it was the main contributor as can be seen in the optimal road paving solution in Figure 1. The Nucleolus solution concept method provides payoffs similar to the Pro Rata method, merely shifting a small amount of payoff from Farm 3 to Farm 1. Comparing the two solution concept methods, the core created by the Shapley Value payoffs was significantly smaller in size than that of the Nucleolus payoffs, indicating that the excess between the payoffs was less and therefore the Shapley Value solution concept method had a fairer distribution of the payoffs.

CONCLUSIONS

Coalition games are being applied to many fields, including economics, financial, computing, politics etc. In this study, efforts have been made to study and understand the level of fairness of 'Solutions Concepts' based on coalition imputations between the two most popular 'Solutions Concepts': Shapley value and Nucleolus used in coalition games. To do that, the computational methods for solution concepts have been presented. In Coalition game, which is a special case of cooperation games derives imputations by implementing few well known solution concepts like Shapley Value, Nucleolus, Core, Kernel etc.

The main contributions of the study:

- A computational method has been proposed to compare and find the level of fairness between Shapley Value and Nucleolus.
- Calculations of imputations using solution concept Nucleolus base on pro-rata based imputations and Shapley Value imputations.
- Generalization of the model for more than three players.
- An example of cost sharing and joint ventures has been presented with coalition model of three players.
- Solution concept Shapley Value has been applied to calculate the worth of coalition.
- Another solution concept namely Core, has been applied to compare the coalition's worth.

In the illustrative example, with barycentric coordinate triangle, Shapley value formed smaller core, Nucleolus values formed a bigger core; so, Shapley value is more fair solution concept, as it formed smaller core. Shapely values have more deviations, i.e. they are fairer compared to Nucleolus based imputations, as they have better distribution than pro-rata distribution. From the Lixicographical orderings, it is difficult to make clear conclusions, as the results were mixed.

REFERENCES

1. Core in a Simple Coalition Formation Game, Social Choice and Welfare, 18-2: 135-153, 2001.
2. Frguson, Thomas, "Part lV- Games in Coalitional Form", in Game Theory, 2nd edition, lecture notes, UCLA., 1014.
3. Fischer, Dietrich, "A Brief Introduction to Game Theory", Technical Report Number 231, PACE University, Ivan G. Seidenberg School of Computer Science and Information Systems, October 2006.
4. Gillies, D. B. (1959). Solutions to general non-zero-sum games. In A. W. Tucker and R. D. Luce : Contributions to the Theory of Games , Princeton University Press, 1959
5. Harsanyi, J. C, A simplified bargaining model for the n-person cooperative game. International Economic Review 4, 1963.
6. Kadiroglu , Atila Abdul, Parag A. Pathak and Alvin A. Roth, PRACTICAL MARKET DESIGN:FOURMATCHES: The New York City High School Match, 2003.

7. Leng, Mimngming Parlar, Mahmut, Analytic Solution for the Nucleolus of a Three-Player Cooperative Game, Department of Computing and Decision Sciences, Lingnan University, 8 Castle Peak Road, Tuen Mun, Hong Kong, October 2009.
8. List of Games, Game Theory, URL: https://en.wikipedia.org/wiki/Cooperative_game_theory
9. Maschler, M. and G. Owen , The consistent Shapley value for games without side payments. Essays in Honor of John Harsanyi. Springer, New York, 1992.
10. Osborne , Martin J., An introduction to Game Theory, 2004, Oxford University Press, ISBN-13:978-0-19512895-6.
11. Osborne , Martin J., Ariel Rubenstein, Business and Economics, 1994.
12. Stengel B. Von, Game Theory, International Programs, University of London, MT3040, 2011.
13. Von Neumann, Oscar Morgenstren, Theory of games and economiv behavior, Princeton, NJ, Princeton University, 1944.
14. Wikipedia, Cooperative Game; https://en.wikipedia.org/wiki/Cooperative_game_theory